Let $D$ be a point on side $BC$ of $\triangle ABC$. Let $I_1$ and $I_2$ be the incenters of $\triangle ABD$ and $ACD$, respectively. Let $K$ be the intersection of lines $BI_2$ and $CI_1$. Prove that $K$ lies on $AD$ if and only if $AD$ bisects angle $A$.

**Solution** (Emma Yang, unedited). We can use barycentric coordinates wrt reference triangle $ABC$. Let $AD = d, CD = m, BD = n$.

Let ray $DI_2$ intersect $\overline{AC}$ at $L$. By the angle bisector theorem, $L$ splits the side $AC$ into an $AD : CD$ ratio, so its barycentric coordinates are $(m : 0 : d)$. Analogously, the barycentric coordinates for $D$ are $(0 : m : n)$.

Let $I_2 = (a : b : s)$ (it is an incenter). Because $D, I_2, L$ are collinear, we can write the following determinant:

$$
\begin{vmatrix}
  m & 0 & d \\
  0 & m & n \\
  a & b & s \\
\end{vmatrix} = 0
$$

If we expand out the determinant and solve for $s$, we get $s = \frac{ad + bn}{m}$. We therefore have:

$$
I_2 = (a : b : \frac{ad + bn}{m}) = (am : bm : ad + bn)
$$

Similarly, we derive:

$$
I_1 = (an : ad + cm : cn)
$$

Since $K$ is the intersection of $BI_2, CI_1$, it must follow the equations of both lines. We can find the coordinates for $K$: 
$K = (amn : m(ad + cm) : n(ad + bn))$

However, we also observe that $K$ is on line $AD$, so we have (due to collinearity):

$$
\begin{vmatrix}
1 & 0 & 0 \\
0 & m & n \\
amn & m(ad + cm) & n(ad + bn)
\end{vmatrix} = 0
$$

This means that:

$$\frac{m}{n} = \frac{m(ad + cm)}{n(ad + bn)}$$

After cross multiplying and simplifying, we obtain $cm = bn$, so after some rearrangement $m : n = b : c$. Using the Angle Bisector Theorem, we see that $D$ indeed is the foot of the angle bisector $\square$

Now, we show that, if $AD$ is the angle bisector, then $K$ lies on $AD$.

Using our previous definitions, we still have $K = (amn : m(ad + cm) : n(ad + bn))$.

Now, we find that the equation of $AD$ is:

$$
\begin{vmatrix}
x & y & z \\
1 & 0 & 0 \\
0 & m & n
\end{vmatrix} = 0
$$

Expanding the given determinant, we have the line equation $-ny + mz = 0$

Furthermore, using the Angle Bisector Theorem, we have $\frac{BD}{DC} = \frac{AB}{AC}$, so we derive:

$$\frac{n}{m} = \frac{c}{b} \Rightarrow bn = cm$$

If we plug in the corresponding $y, z$ values in $K$, we have:

$$-n(m(ad + cm)) + m(n(ad + bn))$$
$$= -mn(ad + cm) + mn(ad + bn)$$
$$= mn(-ad - cm + ad + bn)$$

Noting that the $-ad - cm + ad + bn = 0$:

$$mn(-ad + cm + ad + bn) = 0$$

so $K$ lies on $AD$, as desired $\square$

Having proven both directions, we are done $\blacksquare$
69 (IMO 2012). Given triangle $ABC$ the point $J$ is the center of the excircle opposite the vertex $A$. This excircle is tangent to the side $BC$ at $M$, and to the lines $AB$ and $AC$ at $K$ and $L$, respectively. The lines $LM$ and $BJ$ meet at $F$, and the lines $KM$ and $CJ$ meet at $G$. Let $S$ be the point of intersection of the lines $AF$ and $BC$, and let $T$ be the point of intersection of the lines $AG$ and $BC$. Prove that $M$ is the midpoint of $ST$.

Solution (Emma Yang, Editor: Eric, Image: An). We will present a barycentric coordinates approach.

Using the reference triangle $\triangle ABC$, we define $a = BC$, $b = AC, c = AB$, and $s = \frac{a+b+c}{2}$. Noting that the sum of the barycentric coordinates is $J = \left(\frac{a}{a:b:c}\right)$ (it is the $A$-excenter) $M = \left(0: \frac{s-a}{s-b}: \frac{s-a}{s-c}\right)$, and $K = \left(\frac{s-b}{s-c}: s: 0\right)$, we can write the coordinates of $G$:

From the problem statement, $G$ is the intersection of segments $CJ$ and $KM$, so it satisfies the equations of both lines. Since $G$ is on $CJ$, let $G = \left(-a:b:\frac{as+b+e}{s-b}\right)$. From the diagram, we can see that $G, M, K$ are collinear, so using the formula derived previously:

$$0 = \begin{vmatrix} -a & b & t \\ 0 & s-b & s-c \\ c-s & s & 0 \end{vmatrix}$$

If we expand the determinant, we have:

$$0 = -a(-s(s-c)) - (s-c)(b(s-c) - t(s-b)).$$

Solving the linear equation for $t$ gives $t = \frac{b(s-c) - as}{s-b}$.

We obtain the following:

$$G = \left(-a:b:\frac{-as+b(s-c)}{s-b}\right).$$

Because $T$ lies on $AG$ but also $BC$, it follows that

$$T = \left(0:b:\frac{-as+s-c}{s-b}\right).$$

Multiplying by $(s-b)$, we can simplify this rather complex expression into:

$$T = \left(0:b(s-b): -as+s-c\right).$$

Noticing that $b(s-b) + b(s-c) - as + 0 = -a(s-b)$, we can change these coordinates s.t. they add to 1 to get $T = \left(0, \frac{b}{a}, 1 - \frac{b}{a}\right)$. 
Using the displacement vector \((0, \frac{a}{s}, \frac{b}{s})\), we can use the distance formula to calculate \(MT\):

\[
|MT|^2 = -\frac{a^2 \cdot \frac{s}{a}}{b^2 \cdot \frac{s}{a} + c^2 \cdot \frac{s}{a}}
\]

Simplifying, we have:

\[|MT|^2 = s^2, \text{ so } MT = s\]

We can do similarly for \(MS\) to obtain \(MS = s\) as well, so \(MT = MS\), as desired. 

Comment. just made some minor latex changes
98 (IMO 2014). Points $P$ and $Q$ lie on side $BC$ of acute-angled $\triangle ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points $M$ and $N$ lie on lines $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$, and $Q$ is the midpoint of $AN$. Prove that lines $BM$ and $CN$ intersect on the circumcircle of $\triangle ABC$.

**Solution** (Emma Yang, Kosta, Image: An). We denote the reference triangle's side lengths as $a$, $b$, $c$ for the sides facing $\angle A$, $\angle B$, $\angle C$, respectively. By AA similarity, $\triangle PBA \sim \triangle ABC$. Writing out the corresponding side ratios, we have:

$$\frac{PB}{c} = \frac{c}{a}$$

If we multiply $c$ on both sides of this ratio, we derive that $PB = \frac{c^2}{a}$. Because we know the coordinates of $B = (0 : 1 : 0)$ and $P$ lies on the line $x = 0$, we find that the barycentric coordinates for $P$ are $(0 : 1 - \frac{c^2}{a^2} : \frac{c^2}{a^2})$.

To find the coordinates for $M$, we note that $\triangle MAB$ and $\triangle PAB$ share a height, but since $|AM| = 2|AP|$, $\frac{[MAB]}{[PAB]} = 2$. Similarly, $\frac{[MCB]}{[BCA]} = 2$. Because $\triangle MBP$ and $\triangle ABP$ share a height and their bases are the same length ($MP = AP$), their areas are the same. Similarly, $[APC] = [MPC]$. We therefore have $[MBC] = [MBP] + [MPC] = [ABC]$, but this area is negative.

The barycentric coordinates of $M$ are therefore $(-1 : 2(1 - \frac{c^2}{a^2}) : \frac{2c^2}{a^2})$. We can do similarly for $N$ to find that $N = (-1 : \frac{2b^2}{a^2} : 2(1 - \frac{c^2}{a^2}))$. 
Now that we’ve found the coordinates of $B, M, C, N$, we can find the equations of lines $BM$ and $CN$.

$BM : z = -\frac{2c^2}{a^2}x$

$CN : y = -\frac{2b^2}{a^2}x$, after some rearranging of terms.

Thus $BM \cap CN$, which fulfills the equations of both terms, has the unhomogenized coordinates $(1 : -\frac{2b^2}{a^2} : -\frac{2c^2}{a^2})$. We can easily check this with the equation of a circumcircle ($a^2 yz + b^2 xz + c^2 xy = 0$):

$$a^2(-\frac{2b^2}{a^2})(-\frac{2c^2}{a^2}) + b^2(-\frac{2c^2}{a^2}) + c^2(-\frac{2b^2}{a^2}) = \frac{4b^2c^2}{a^2} - \frac{2b^2c^2}{a^2} - \frac{2b^2c^2}{a^2} = 0$$

Solution. (Jaemin, unedited)

Let us draw a parallelogram $BCB'C'$ such that $A$ is the intersection of diagonals $C'C$ and $B'B$.

Claim: $\triangle BCC'$ is similar to $\triangle BAM$

Proof: First of all, we know that $\triangle ABC$ is similar to $\triangle PBA$. Now point $C'$ is basically the reflection of $C$ over $A$ while point $M$ is the reflection of $A$ over $P$ (or we can see this by $SAS$ similarity as we still preserve the ratio of sides if we multiply a corresponding side of both triangles by a common ratio).

Similarly, we can derive that $\triangle CBB'$ is similar to $\triangle CAN$.

Now, if we wish to prove that the intersection of $CN$ and $BM$ lies on the circumcircle of $\triangle ABC$ (let’s denote this point as $D$), then we need to prove that $\angle ABM + \angle ACN = 180^\circ$ as that would prove that $ABDC$ is a cyclic quadrilateral.

Now this is equivalent to proving that $\angle C'BC + \angle BCB' = 180^\circ$. This follows directly from our construction as $BCB'C'$ is a parallelogram and adjacent angles in a parallelogram add up to $180^\circ$. ■

(currently diagramless)