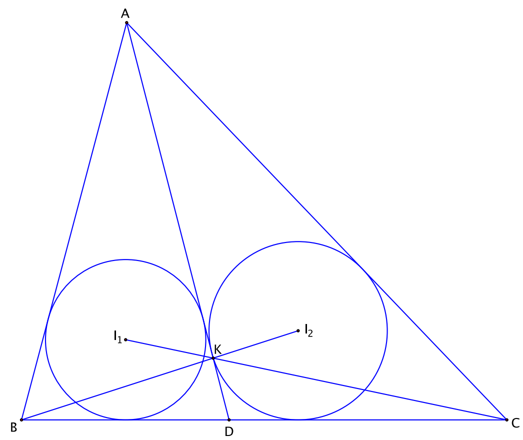
135 (Evan Chen)**.** Let D be a point on side BC of $\triangle ABC$. Let I_1 and I_2 be the incenters of $\triangle ABD$ and ACD, respectively. Let K be the intersection of lines BI_2 and CI_1. Prove that K lies on AD if and only if AD bisects angle A.



Solution (Emma Yang, unedited)**.** We can use barycentric coordinates wrt reference triangle ABC. Let $AD = d, CD = m, BD = n$.

Let ray DI_2 intersect $\overline{AC}$ at L. By the angle bisector theorem, L splits the side AC into an $AD : CD$ ratio, so its barycentric coordinates are $(m : 0 : d)$. Analogously, the barycentric coordinates for D are $(0 : m : n)$.

Let $I_2 = (a : b : s)$ (it is an incenter). Because D, I_2, L are collinear, we can write the following determinant:

$$\begin{vmatrix} m & 0 & d \\ 0 & m & n \\ a & b & s \end{vmatrix} = 0$$

If we expand out the determinant and solve for s, we get $s = \frac{ad+bn}{m}$. We therefore have:

$$I_2 = (a : b : \frac{ad+bn}{m}) = (am : bm : ad+bn)$$

.

Similarly, we derive:

$$I_1 = (an : ad+cm : cn)$$

Since K is the intersection of BI_2, CI_1, it must follow the equations of both lines. We can find the coordinates for K:

$$K = (amn : m(ad + cm) : n(ad + bn)$$

However, we also observe that K is on line AD, so we have (due to collinearity):

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & m & n \\ amn & m(ad+cm) & n(ad+bn) \end{vmatrix} = 0$$

This means that:

$$\frac{m}{n} = \frac{m(ad+cm)}{n(ad+bn)}$$

.

After cross multiplying and simplifying, we obtain $cm = bn$, so after some rearrangement $m : n = b : c$. Using the Angle Bisector Theorem, we see that D indeed is the foot of the angle bisector □

Now, we show that, if AD is the angle bisector, then K lies on AD.

Using our previous definitions, we still have $K = (amn : m(ad + cm) : n(ad + bn)))$.

Now, we find that the equation of AD is:

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & m & n \end{vmatrix} = 0$$

Expanding the given determinant, we have the line equation $-ny + mz = 0$

Furthermore, using the Angle Bisector Theorem, we have $\frac{BD}{DC} = \frac{AB}{AC}$, so we derive:

$$\frac{n}{m} = \frac{c}{b} \Rightarrow bn = cm$$

If we plug in the corresponding y, z values in K, we have:

$-n(m(ad + cm)) + m(n(ad + bn))$
$= -mn(ad + cm) + mn(ad + bn)$
$= mn(-ad - cm + ad + bn)$

Noting that the $-ad - cm + ad + bn = 0$:
$mn(-ad + cm + ad + bn) = 0$, so K lies on AD, as desired □

Having proven both directions, we are done ■

69 (IMO 2012)**.** Given triangle ABC the point J is the center of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

Solution (Emma Yang, Editor: Eric, Image: An)**.** We will present a barycentric coordinates approach.

Using the reference triangle $\triangle ABC$, we define $a = BC$, $b = AC$,$c = AB$, and $s = \frac{a+b+c}{2}$. Noting that the sum of the barycentric coordinates is $J = (-a : b : c)$ (it is the A-excenter) $M = (0 : s-b : s-c)$, and $K = (-(s-c) : s : 0)$, we can write the coordinates of G:

From the problem statement, G is the intersection of segments CJ and KM, so it satisfies the equations of both lines. Since G is on CJ, let $G = (-a, b, t)$. From the diagram, we can see that G, M, K are collinear, so using the formula derived previously:

$$0 = \begin{vmatrix} -a & b & t \\ 0 & s-b & s-c \\ c-s & s & 0 \end{vmatrix}$$

If we expand the determinant, we have:
$0 = -a(-s(s-c)) - (s-c)(b(s-c) - t(s-b))$. Solving the linear equation for t gives $t = \frac{b(s-c)-as}{s-b}$

We obtain the following:

$$G = (-a : b : \frac{-as + b(s-c)}{s-b})$$

.

Because T lies on AG but also BC, it follows that

$$T = (0 : b : \frac{-as + s - c}{s-b})$$

Multiplying by $(s-b)$, we can simplify this rather complex expression into:

$$T = (0 : b(s-b) : -as + s - c)$$

Noticing that $b(s-b) + b(s-c) - as + 0 = -a(s-b)$, we can change these coordinates s.t. they add to 1 to get $T = (0, \frac{b}{a}, 1 - \frac{b}{a})$.

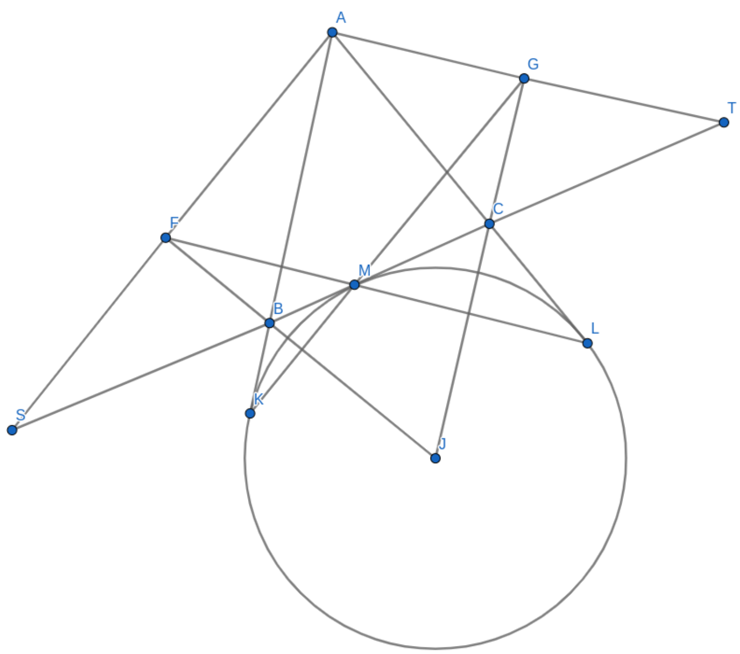
Using the displacement vector $(0, \frac{s}{a}, \frac{-s}{a})$, we can use the distance formula to calculate MT:

$$|MT|^2 = -a^2 \cdot \frac{s}{a} \cdot \frac{-s}{a} - b^2 \cdot \frac{-s}{a} \cdot 0 - c^2 \cdot 0 \cdot \frac{s}{a}$$

Simplifying, we have:

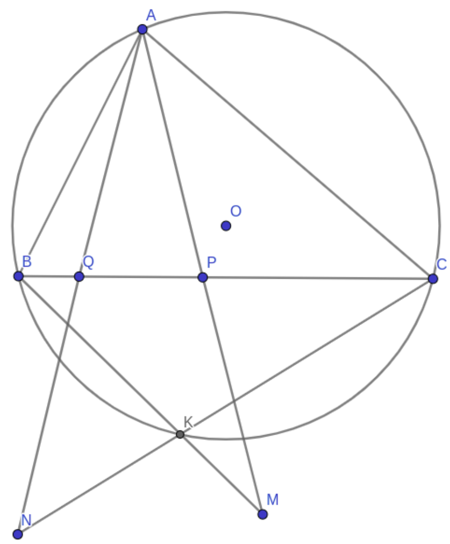
$|MT|^2 = s^2$, so $MT = s$

We can do similarly for MS to obtain $MS = s$ as well, so $MT = MS$, as desired ■



Comment. just made some minor latex changes

98 (IMO 2014)**.** Points P and Q lie on side BC of acute-angled $\triangle ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of $\triangle ABC$.



Solution (Emma Yang, Kosta, Image: An)**.** We denote the reference triangle's side lengths as a, b, c for the sides facing $\angle A$, $\angle B$, $\angle C$, respectively.

By AA similarity, $\triangle PBA \sim \triangle ABC$. Writing out the corresponding side ratios, we have:

$\frac{PB}{c} = \frac{c}{a}$

If we multiply c on both sides of this ratio, we derive that $PB = \frac{c^2}{a}$. Because we know the coordinates of $B = (0:1:0)$ and P lies on the line $x = 0$, we find that the barycentric coordinates for P are $(0:1-\frac{c^2}{a^2}:\frac{c^2}{a^2})$

To find the coordinates for M, we note that $\triangle MAB$ and $\triangle PAB$ share a height, but since $|AM| = 2|AP|$, $\frac{[MAB]}{[PAB]} = 2$. Similarly, $\frac{[MCA]}{[BCA]} = 2$.

Because $\triangle MBP$ and $\triangle ABP$ share a height and their bases are the same length ($MP = AP$), their areas are the same. Similarly, $[APC] = [MPC]$. We therefore have $[MBC] = [MBP] + [MPC] = [ABC]$, but this area is negative.

The barycentric coordinates of M are therefore $(-1:2(1-\frac{c^2}{a^2}):\frac{2c^2}{a^2})$. We can do similarly for N to find that $N = (-1:\frac{2b^2}{a^2}:2(1-\frac{c^2}{a^2}))$.

Now that we've found the coordinates of B, M, C, N, we can find the equations of lines BM and CN.

$BM: z = -\frac{2c^2x}{a^2}$

$CN: y = -\frac{2b^2x}{a^2}$, after some rearranging of terms.

Thus $\overline{BM} \cap \overline{CN}$, which fulfills the equations of both terms, has the unhomogenized coordinates $(1 : -\frac{2b^2}{a^2} : -\frac{2c^2}{a^2})$. We can easily check this with the equation of a circumcircle $(a^2yz + b^2xz + c^2xy = 0)$:

$a^2(-\frac{2b^2}{a^2})(-\frac{2c^2}{a^2}) + b^2(-\frac{2c^2}{a^2}) + c^2(-\frac{2b^2}{a^2}) = \frac{4b^2c^2}{a^2} - \frac{2b^2c^2}{a^2} - \frac{2b^2c^2}{a^2} = 0$ ■

Solution. (Jaemin, unedited)

Let us draw a parallelogram $BCB'C'$ such that A is the intersection of diagonals $C'C$ and $B'B$.

Claim: $\triangle BCC'$ is similar to $\triangle BAM$

Proof: First of all, we know that $\triangle ABC$ is similar to $\triangle PBA$. Now point C' is basically the reflection of C over A while point M is the reflection of A over P (or we can see this by SAS similarity as we still preserve the ratio of sides if we multiply a corresponding side of both triangles by a common ratio)

Similarly, we can derive that $\triangle CBB'$ is similar to $\triangle CAN$.

Now, if we wish to prove that the intersection of CN and BM lies on the circumcircle of $\triangle ABC$ (let's denote this point as D), then we need to prove that $\angle ABM + \angle ACN = 180°$ as that would prove that $ABDC$ is a cyclic quadrilateral.

Now this is equivalent to proving that $\angle C'BC + \angle BCB' = 180°$. This follows directly from our construction as $BCB'C'$ is a parallelogram and adjacent angles in a parallelogram add up to 180°. ■

(currently diagramless)