Day 9 homework

Please do ONE problem (or more if you like to):

1. Misha and Serina tried to meet for lunch (using the conditions in Problem 1), but they failed! Misha waited for 15 minutes and no one came to lunch with him. What is the probability that if he had waited another 15 minutes, they would have eaten lunch together?

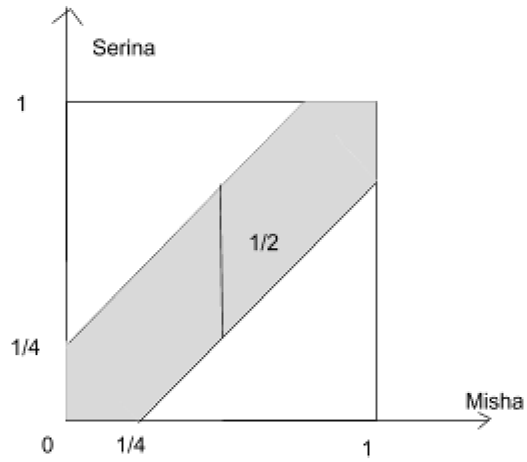
2. Positive numbers a, b, and c are chosen so that $a + b + c = 1$. What is the probability that $a^2 + b^2 < c^2$?

3. A rod is cut into three segments. What is the probability that these three parts can form a triangle?

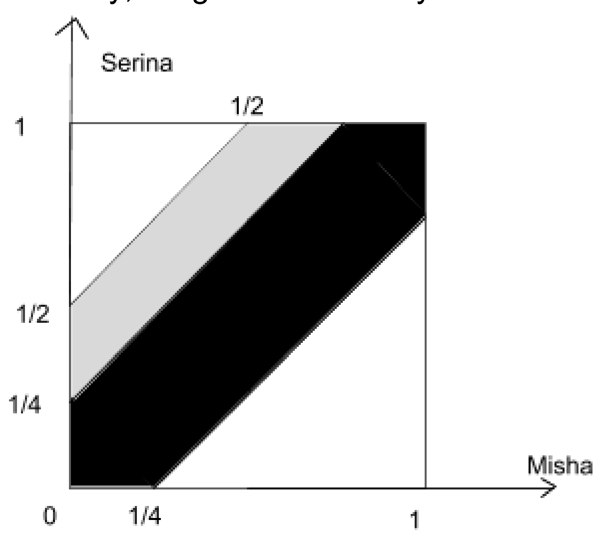
4. Finish the proof of #3 from class: Variables x, y, and z are chosen in the interval $[0, 1]$. Find the probability that no two of them are less than $\frac{1}{4}$ apart.

1.

We can create a graph with the x axis as the time in which Misha arrives and the y axis as the time in which Serina arrives. We know that in the case of them each waiting 15 minutes, we have the following graph, with the shaded region as the time in which they can arrive and meet up.



Since we know in the new condition that they were unable to meet up, the shaded region in the above graph would be unshaded. The probability of Misha meeting Serina after waiting 15 minutes more would be the new shaded region bounded by the lines (0,¼) to (¾,1) and (0,½) to (½,1). Since by waiting 15 more minutes, he would extend the region in which Serina could arrive by ¼ so by creating a line parallel to his original boundary, we get the boundary of the new condition.



The area of the shaded region is 5/32. Since probability is calculated by favorable over total, the probability would be (5/32)/(9/16). The total is 9/16, the area of the square minus the dark region, because we know they are unable to meet up in the first condition, so we are definitely not in the dark region. Therefore the probability is 5/18.

Evan Song, Day 9 Homework

1.

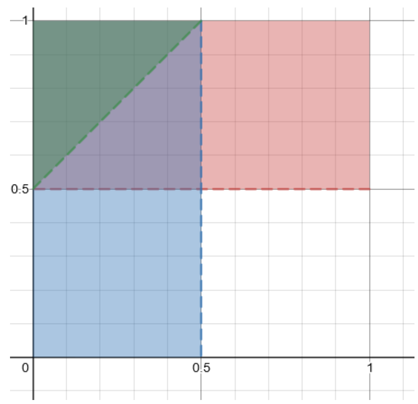
3. Since the line is cut into 3 parts, it must be cut at two points. Let's call these points x and y. If we let the length of the entire line be 1, then the lengths of the 3 parts must be x, $y - x$, and $1 - x$. These lengths must fulfill the Triangle Inequality, so we have:

$$x > \frac{1}{2}$$

$$x + \frac{1}{2} > y$$

$$y < \frac{1}{2}$$

Our total possibilities can be represented with a 1×1 square, since the maximum value of x and y is 1 and their minimum value is 0. Graphing the inequalities in the square yields the following figure:

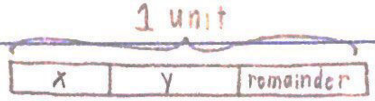


The top left triangle (dark green) represents our desired outcomes, but since x and y are interchangeable, we have another triangle, in the bottom right corner. We see both triangles have area $\frac{1}{8}$, so our total probability is $\frac{\frac{1}{4}}{1} = \frac{1}{4}$.

Stephanie Shen
NYMC Blue PM
July 30, 2020
Homework #9

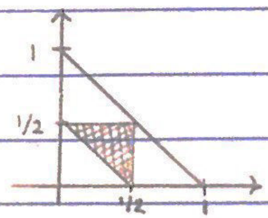
3. Rule → The sum of the lengths of 2 sides in a △ must be > third side.

① WLOG → you have a rod of length 1 unit. It's impossible that the length of one side is $\geq \frac{1}{2}$ or else the rule is broken.



② conclude that $x+y > \frac{1}{2}$, $x < \frac{1}{2}$, and $y < \frac{1}{2}$

③ graph for geometric probability



Probability is 1/4

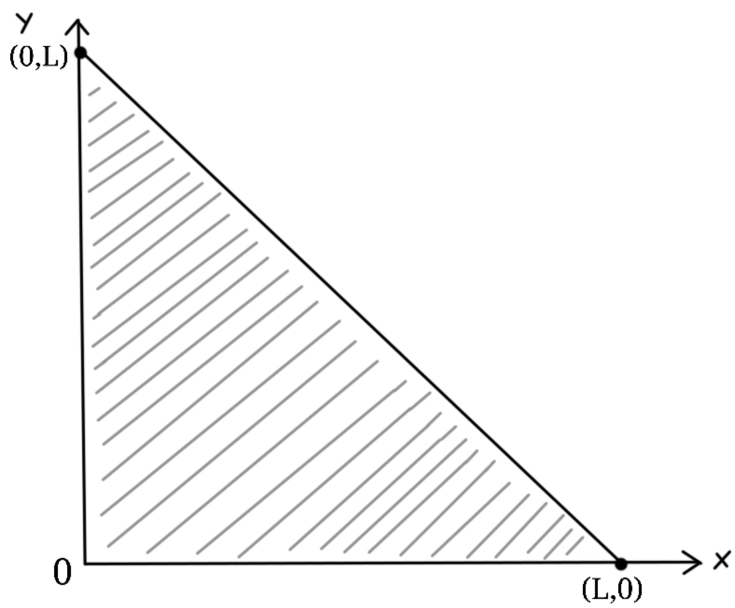
Math Circle Day 9 Homework

Anastasia Lee

July 30, 2020

My Solution to Problem #3:

Let L be the length of the rod, and let x, y, and $L - x - y$ be the lengths of the three segments. The total possible ways of breaking the rod into three parts can be represented by the area of the triangle formed by the inequalities $x > 0$, $y > 0$, and $x + y < L$.



However, in order for the three pieces to form a triangle, the Triangle Inequality must be satisfied, so we have

$$(1)\ \ x + y > L - x - y$$

$$2x + 2y > L$$

$$x + y > \frac{L}{2}$$

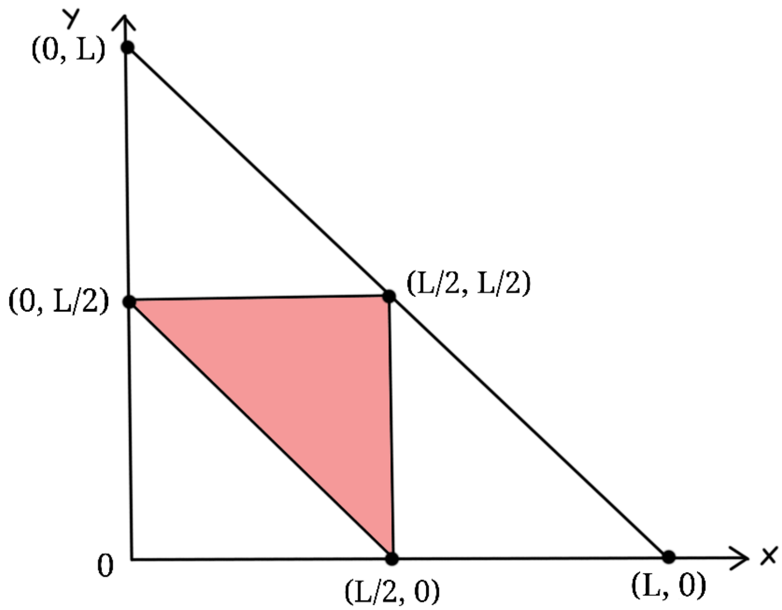
$$(2)\ \ x + (L - x - y) > y$$

$$L - y > y$$

$$y < \frac{L}{2}$$

$$
\begin{gathered}
(3)\;\; y+(L-x-y)>x \\
L-x>x \\
x<\frac{L}{2}.
\end{gathered}
$$

Finding the intersection of the graphs of these three inequalities tells us that the successful region can be represented as the area of the shaded triangle:

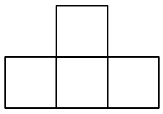


Since the shaded triangle is similar to the larger original triangle with the ratio of their side lengths being $1:2$, the ratio of their areas will be $1:4$. Therefore, the probability that a triangle can be formed from the three segments is $\boxed{\frac{1}{4}}$.

Day 11 homework

Please do ONE problem (or more if you like to):

1. A *camel* is a new chess piece that travels 4 squares in one direction and 1 square over, as opposed to a knight that travels 3 squares in one direction and 1 square over. Is it possible for a camel to start in one square of a 10×10 board and visit every square precisely once?
2. Prove that a 10×10 board can not be divided into T-tetrominoes. Below is an example of a T-tetromino:



3. We are given 3D tiles which are formed by attaching unit cubes to a given unit cube along three different faces so that all 4 cubes share a vertex. Is it possible to fill a $11 \times 12 \times 13$ box completely without any missing cubes or any cubes sticking out?

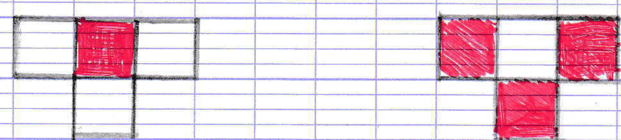
Julius Gross

Day 11 Homework

2. 10×10 board $\longrightarrow$ 100 squares $\longrightarrow$ by coloring: 50 red squares and 50 white squares
   1 T-tetromino $\longrightarrow$ 4 squares

   $\downarrow$

   by coloring we can have either 1 red square and 3 white squares or 3 red squares and 1 white square:



one 10×10 board will be divided into 25 T-tetrominoes $(4 \times 25 = 100)$, if it is possible. However, with 25 T-tetrominoes will not have 50 red squares and 50 white squares, but either 25 red squares and 75 white squares or 75 red squares and 25 white squares. Thus, it is impossible that a 10×10 board can be divided into T-tetrominoes.

Math Circle Trigonometry Solutions

Anastasia Lee

July 26, 2020

Problem #2:

Since A is an acute angle, we know that it is in the first quadrant and that $\sin A$ and $\cos A$ are both positive. Thus, $\sin A + \cos A$ is also positive. Then, I recognized 169 as 13^2, so I knew that the hypotenuse of the right triangle was 13. I am familiar with the 5-12-13 Pythagorean triple, so I saw that $\sin A$ and $\cos A$ were $\frac{5}{13}$ and $\frac{12}{13}$, in some order. Therefore, $\sin A + \cos A = \boxed{\dfrac{17}{13}}$.

Problem #3:

Since $135° \leq x \leq 180°$, we know that $\sin x$ will be positive, and $\cos x$ will be negative. After seeing that $\sin x + \cos x = -\frac{1}{5}$, I thought of the 3-4-5 Pythagorean triple, which I noticed would work since $\frac{3}{5} - \frac{4}{5} = -\frac{1}{5}$. Thus, $\sin x = \frac{3}{5}$ and $\cos x = -\frac{4}{5}$. Then, we know that

$$\cos 2x = \cos^2 x - \sin^2 x = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} = \frac{9}{25} = \boxed{\frac{7}{25}}.$$

Problem #4:

Since x is in the second quadrant, we know that $\sin x$ will be positive, and $\cos x$ will be negative. This time, $\sin x$ is $\frac{1}{5}$ larger than $\cos x$, so using my knowledge of the 3-4-5 Pythagorean triple, $\sin x = \frac{4}{5}$ and $\cos x = -\frac{3}{5}$. Then, since $\tan x = \frac{\sin x}{\cos x}$, we have

$$\tan x = \frac{\frac{4}{5}}{-\frac{3}{5}} = \frac{4}{-3} = \boxed{-\frac{4}{3}}.$$

Problem #6:

We can begin by rewriting the fraction as

$$\frac{\sin 40° + \sin 80°}{\sin 50° + \sin 10°},$$

since $\cos x = \sin 90 - x$. Then, we can compute the numerator and denominator separately, using the fact that

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}.$$

We have:

$$\sin 40^\circ + \sin 80^\circ = 2\sin\frac{120^\circ}{2}\cos\frac{-40^\circ}{2} = 2\sin 60^\circ \cos -20^\circ = 2\sin 60^\circ \cos 20^\circ.$$

Similarly,

$$\sin 50^\circ + \sin 10^\circ = 2\sin\frac{60^\circ}{2}\cos\frac{40^\circ}{2} = 2\sin 30^\circ \cos 20^\circ.$$

Thus,

$$\frac{\sin 40^\circ + \sin 80^\circ}{\sin 50^\circ + \sin 10^\circ} = \frac{2\sin 60^\circ \cos 20^\circ}{2\sin 30^\circ \cos 20^\circ} = \frac{\sin 60^\circ}{\sin 30^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \boxed{\sqrt{3}}.$$

Problem #7:

We know that

$$\sin 2x = 2\sin x \cos x,$$

so we have

$$2\sin x \cos x = \frac{24}{25}$$

or

$$\sin x \cos x = \frac{12}{25}.$$

Since $\sqrt{25} = 5$, the hypotenuse of the right triangle is 5, and the 3-4-5 Pythagorean triple helps us realize that $\sin x$ and $\cos x$ are $\pm\frac{3}{5}$ and $\pm\frac{4}{5}$, in some order. Thus,

$$\sin^4 x + \cos^4 x = \left(\pm\frac{3}{5}\right)^4 + \left(\pm\frac{4}{5}\right)^4 = \frac{81}{625} + \frac{256}{625} = \boxed{\frac{337}{625}}.$$

Problem #8:

First, we can factor $\sin^3 x + \cos^3 x$ into

$$(\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x).$$

Then, we know that $\sin x + \cos x = \frac{1}{2}$ and that $\sin^2 x + \cos^2 x = 1$, so we can rewrite it as

$$\frac{1}{2}(1 - \sin x \cos x).$$

Now, we can find the value of $\sin x \cos x$ by squaring $(\sin x + \cos x)$:

$$\begin{aligned}
(\sin x + \cos x)^2 &= \frac{1}{4} \\
\sin^2 x + 2\sin x \cos x + \cos^2 x &= \frac{1}{4} \\
1 + 2\sin x \cos x &= \frac{1}{4} \\
2\sin x \cos x &= -\frac{3}{4} \\
\sin x \cos x &= -\frac{3}{8}.
\end{aligned}$$

Therefore, we have

$$\frac{1}{2}(1 - \sin\, x\, \cos\, x) = \frac{1}{2}(1 - \left(-\frac{3}{8}\right)) = \frac{1}{2}\left(\frac{11}{8}\right) = \boxed{\frac{11}{16}}.$$

New York Math Circle Summer Program 2020
HSB Siwanowicz
7/23/2020

Rin Fukuoka

Please submit solutions to at least 5 problems by 4:00 pm 7/27/20 on Google Classroom. Answers and solutions will be posted shortly after that time.

Problems from Class

2. Acute angle A satisfies $(\sin A)(\cos A) = 60/169$. Find $\sin A + \cos A$.

$$\sin A \cos A = \frac{60}{169}$$

$$2 \sin A \cos A = \frac{120}{169}$$

Adding $\sin^2 A + \cos^2 A$ to both sides,

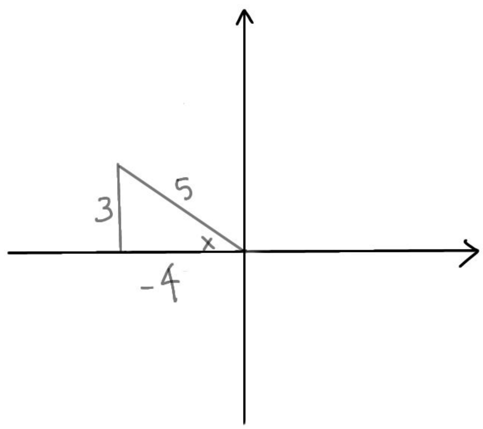
$$\sin^2 A + 2 \sin A \cos A + \cos^2 A = \frac{120}{169} + \sin^2 A + \cos^2 A$$

$$\sin^2 A + 2 \sin A \cos A + \cos^2 A = \frac{120}{169} + 1$$

$$(\sin A + \cos A)^2 = \frac{289}{169}$$

$$\sin A + \cos A = \boxed{\pm \frac{17}{13}}$$

3. If $\sin x + \cos x = -\frac{1}{5}$, and $135° \leq x \leq 180°$, find the value of $\cos 2x$.



$-\frac{1}{5} = \frac{3}{5} - \frac{4}{5}$. Realize that 3-4-5 right Δ satisfies the equation.

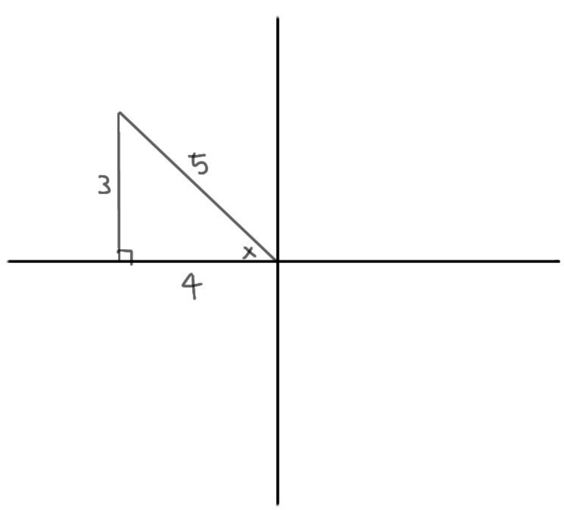
So $\sin x = \frac{3}{5}$, $\cos x = -\frac{4}{5}$.

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2$

$\cos 2x = \boxed{\frac{7}{16}}$

4. Find the value of $\tan x$ if $\sin x + \cos x = \frac{1}{5}$ and x is in the second quadrant.



We realize that $\frac{4}{5} - \frac{3}{5} = \frac{1}{5}$.

Either $\sin x = \frac{4}{5}$ and $\cos x = -\frac{3}{5}$

OR

$\sin x = \frac{3}{5}$ and $\cos x = -\frac{4}{5}$.

Since x is in the second quadrant, $\cos x$ must be negative.

So $\sin x = \frac{3}{5}$, an $\cos x = -\frac{4}{5}$.

$\tan x = \frac{\sin x}{\cos x}$

$\tan x = \frac{\left(\frac{3}{5}\right)}{\left(-\frac{4}{5}\right)}$

$\tan x = \boxed{-\frac{3}{4}}$

5. If $\cos 3x + \cos x = 0$, find all possible values of $\cos x$

$$\cos 3x + \cos x = 0$$

$$2\cos\frac{(3x+x)}{2}\cos\frac{(3x-x)}{2} = 0$$

$$2(\cos 2x)(\cos x) = 0$$

$$2(2\cos^2 x - 1)(\cos x) = 0$$

$$(2\cos^2 x - 1)(\cos x) = 0$$

$$\cos x = \boxed{\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1}$$

What is the value of x?

ⓐ $2\cos^2 x - 1 = 0$

$$\cos^2 x = \frac{1}{2}$$

$$\cos x = \pm\frac{\sqrt{2}}{2}$$

$$x = \frac{\pi}{4}, \frac{5}{4}\pi \ldots$$

$$x = \begin{cases} \frac{\pi}{4} + 2\pi a \\ \frac{5}{4}\pi + 2\pi a \end{cases}$$

For any integer a.

ⓑ $\cos x = 0$

$$x = \frac{\pi}{2}, \frac{3}{2}\pi \ldots$$

$$x = \begin{cases} \frac{\pi}{2} + 2\pi b \\ \frac{3}{2}\pi + 2\pi b \end{cases}$$

For any integer b.

6. Express in simplest form: $\frac{\sin 40° + \sin 80°}{\cos 40° + \cos 80°}$

$$\frac{\sin 40° + \sin 80°}{\cos 40° + \cos 80°}$$

$$= \frac{2\sin\frac{80°+40°}{2}\cos\frac{80°-40°}{2}}{2\cos\frac{80°+40}{2}\cos\frac{80°-40°}{2}}$$

$$= \frac{\cancel{2}\sin 60°\ \cancel{\cos 20°}}{\cancel{2}\cos 60°\ \cancel{\cos 20°}}$$

$$= \frac{\sin 60°}{\cos 60°}$$

$$= \tan 60°$$

$$= \boxed{\sqrt{3}}$$

7. If $\sin 2x = \frac{24}{25}$, find the value of $\sin^4 x + \cos^4 x$

1. $\sin 2x = \frac{24}{25}$

$2\sin x\cos x = \frac{24}{25}$

$\sin x\cos x = \frac{12}{25}$

2. $\sin^2 x + \cos^2 x = 1$

$(\sin^2 x + \cos^2 x)^2 = 1$

$\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x = 1$

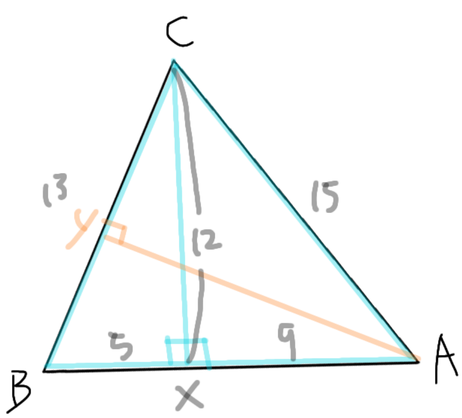
$\sin^4 x + \cos^4 x = 1 - 2\sin^2 x\cos^2 x$

$\sin^4 x + \cos^4 x = 1 - 2(\sin x\cos x)^2$

$\sin^4 x + \cos^4 x = 1 - 2\left(\frac{12}{25}\right)$

$\sin^4 x + \cos^4 x = \frac{337}{625}$

9. In triangle ABC, $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$. Find $\tan C$.



$\tan C = \frac{AY}{CY}$.

Let $BC = 13$.

$\cos B = \frac{5}{13} = \frac{BX}{BC}$, So $BX = 5$.

By the Pythagorean theorem, $CX = 12$.

Since $\sin A = \frac{4}{5} = \frac{CX}{AC}$, $AC = 15$.

By the Pythagorean theorem, $AX = 9$.

$[ABC] = \frac{12(5+9)}{2} = 84$

$[ABC] = \frac{AY \cdot 13}{2} = 84$

$AY = \frac{168}{13} \ldots ①$

$CY = \sqrt{15^2 - \frac{168^2}{13^2}} = \frac{99}{13} \ldots ②$

(Continued to next page)

$$\tan C = \frac{AY}{CY}$$

From ① and ②, $\tan C = \frac{\left(\frac{168}{13}\right)}{\left(\frac{99}{13}\right)}$

$$\tan C = \boxed{\frac{56}{33}}$$

11. If $\sin^6 x + \cos^6 x = \frac{2}{3}$, Find all possible values of $\sin 2x$

$$\sin^2 x + \cos^2 x = 1$$

$$(\sin^2 x + \cos^2 x)^3 = 1^3$$

$$\sin^6 x + \cos^6 x + 3\sin^4 \cos^2 x + 3\cos^4 x \sin^2 x = 1$$

$$\sin^6 x + \cos^6 x + 3(\sin^2 x \cos^2 x)(\sin^2 x + \cos^2 x) = 1$$

$$3(\sin^2 x \cos^2 x)(\sin^2 x + \cos^2 x) = 1 - \sin^6 x \cos^6 x$$

$$3(\sin 2x)^2 \cdot (1) = 1 - \left(\frac{2}{3}\right)$$

$$3(\sin 2x)^2 = \frac{1}{3}$$

$$\sin^2 2x = \frac{1}{9}$$

$$\sin 2x = \boxed{\pm\frac{1}{3}}$$