1. A sphere is inscribed in a cone, and a cylinder is circumscribed about the sphere so that the bases of the cone and cylinder are in the same plane. Let V_1 be the volume of the cone, and let V_2 be the volume of the cylinder.

   (a) Prove that $V_1 = V_2$ is impossible.

   (b) Find the smallest k for which $V_1 = kV_2$, and in this case determine the angle at the vertex of the cone.

2. (a) In a group of six people, everyone either knows each other or doesnt know each other. Show there is a group of three people that either all know each other or all dont know each other.

   (b) Seventeen people correspond by mail with one another - each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

3. (a) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$, where a, b and c are real numbers.

   (b) Find the maximum value of the constant k so that the inequality

$$a^2 + b^2 + c^2 + d^2 \geq k(ab + ac + ad + bc + bd + cd)$$

   holds for all real numbers a, b, c, and d.

   (c) Generalize these results.

4. There are nonzero integers a, b, r, and s such that the complex number $r + si$ is a zero of the polynomial $P(x) = x^3 - ax^2 + bx - 65$. For each possible combination of a and b, let $p_{a,b}$ be the sum of the zeroes of $P(x)$. Find the sum of the $p_{a,b}s$ for all possible combinations of a and b.

5. Let A = {1, 2, 3, 4, 5, 6, 7} and let N be the number of functions f from set A to set A such that $f(f(x))$ is a constant function. Find the remainder when N is divided by 1000.

6. Point C is on $\overline{AB}$. An arbelos is formed by erecting semicircles on the same side of line AB with diameters $\overline{AB}$, $\overline{AC}$ and $\overline{CB}$. Circle k_1 is tangent to the three semicircles of the arbelos. Let O_1 denote the center of $k1$, let $d1$ be the diameter of $k1$, and let $h1$ be the distance from $O1$ to $\overline{AB}$. Show that $d_1 = h_1$.
   Circle k_2 is tangent to $k1$ and to the semicircles with diameters $\overline{AB}$ and $\overline{AC}$. What can you conclude about d_2 and h_2 (where d_2 and h_2 are defined as above)?
   Generalize.

7. Let S be the set of all polynomials of the form $z^3 + az^2 + bz + c$, where a, b, and c are integers. Find the number of polynomials in S such that each of its roots z satisfies either $|z| = 20$ or

$|z| = 13$.

8. Let p and q be natural numbers such that $\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}$. Prove that p is divisible by 1979.

9. Circles k_1, k_2, k_3, k_4 intersect as follows: $k_1 \cap k_2 = \{A_1, A_2\}$, $k_2 \cap k_3 = \{B_1, B_2\}$, $k_3 \cap k_4 = \{C_1, C_2\}$, and $k_4 \cap k_1 = \{D_1, D_2\}$. Prove
   (a) If A_1, B_1, C_1, D_1, are collinear/concyclic, then so are A_2, B_2, C_2, D_2.
   (b) If A_1, A_2, C_1, C_2, are collinear/concyclic, then so are B_1, B_2, D_1, D_2.

10. (a) Express 2002^{2002} as the sum of four cubes.
   (b) Determine the smallest positive integer k such that there exist integers $x_1, x_2, \ldots, x_k$ with $x_1^3 + x_2^3 + \cdots x_k^3 = 2002^{2002}$.

11. Consider the sequence $a_1, a_2, \cdots$ defined by $a_n = 2_n + 3_n + 6_n - 1$ for all positive integers n. Determine all positive integers that are relatively prime to every term of the sequence.

12. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

13. Prove that the roots of $x^5 + ax^4 + bx^3 + cx^2 + d^x + e = 0$ cannot all be real if $2a^2 < 5b$. In the following three problems, let $\omega = \text{cis}\frac{2\pi}{n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, where n is a positive integer.

14. (a) Prove that $1 + \text{cis}\theta = 2\cos\frac{\theta}{2}\text{cis}\frac{\theta}{2}$.
   (b) Prove that $1 - \text{cis}\theta = -2i\sin\frac{\theta}{2}\text{cis}\frac{\theta}{2}$.

15. (a) Express $z^{n-1} + z^{n-2} + \cdots + 1$ as a product of linear factors.
   (b) Prove that the product $\prod_{k=1}^{n-1}(1 - \omega^k)$ is equal to n.
   (c) If n is odd, then prove that $\prod_{k=1}^{n}(1 + \omega^k) = 2$. What if n is even?

16. (a) Show that $1 - \omega^k = -2i\sin\left(\frac{k\pi}{n}\right)\text{cis}\left(\frac{k\pi}{n}\right)$.

(b) Use 15b to show that $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}$. [Recall that the magnitude of a product of complex numbers equals the product of their magnitudes.]

(c) Evaluate $\sin 5^\circ \sin 10^\circ \sin 15^\circ \sin 175^\circ$.

(Level C Sample Problems 4, 5, and 7 are from the 2013 AIME I and II. Problems 8, 10b, 11, 12, and 13, are from the 1979 IMO, 2003 IMO shortlist, 2005 IMO, 1986 IMO, and 1983 USAMO.)