- Levels Green, Blue and Black are graded separately.

- The exam consists of short questions and long questions.

- You may answer questions in multiple levels - we recommend you choose two, but please complete all questions in at least one level to enhance your chances of being admitted.

- Regardless of the number of levels you attempt, you must complete this exam within the two-hour window. There will be an additional12-minute grace period for printing, scanning and emailing the test to info+exam@nymathcircle.org. If you don't have a scanner, you may email a picture to us.

- A timer is started once you complete the Google Form - you will receive occasional e-mail reminders on time remaining. The time stamp on your submitted exam is proof of time taken on the exam.

- You will be admitted to the highest you qualify for.

- **Sign the statement at the end of this document.**

use for scratch:

use for scratch:

Short Answer: Write your answer and show your work in the space below each question. Clearly indicate your final answer by drawing a $\boxed{\text{box}}$ around it.

1. Simplify the expression $2x - 3(x-2) + (x - 2(2x-1))$.

$$2x - 3(x-2) + (x - 2(2x-1)) = \boxed{-4x+8}$$

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2. What is the area of an isosceles trapezoid with sides 5, 4, 5, and 10?

height is 4 so area is $\boxed{28}$

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3. If $(x+a)^2 = 49$ and $(x+b)^2 = 9$, write down **all** possible values of $a - b$.

$$x + a = \pm 7$$
$$x + b = \pm 3$$

so values are $\boxed{\pm 4 \text{ and } \pm 10}$

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4. Patricia bicycled from City A to City B at an average rate of 12 miles per hour (mph) and returned over exactly the same route from City B to City A at an average rate of 16 mph. The entire trip took 7 hours. Find the distance from City A to City B. Let t be the time to go from A to B. Then

$$\begin{aligned} 12t &= 16(7-t) \\ 28t &= 112 \\ t &= 4 \end{aligned}$$

so the distance from A to B is $\boxed{48 \text{ miles}}$

5. A convoy of trucks carrying 2800 lbs. each started to cross a desert. When two trucks broke down and had to be abandoned their cargo was distributed equally between the other trucks, resulting in each working truck carrying 224 lbs. more. How many trucks were in the original convoy? Let n be the number of trucks in the original convoy. Then

$$\begin{aligned} 2800n &= 3024(n-2) \\ 224n &= 6048 \\ n &= \boxed{27} \end{aligned}$$

6. Compute the number of teams consisting of 3 boys and 2 girls that can be formed if there are 4 boys and 5 girls from which to select, and Robert (one of the boys) must be on the team. $\binom{3}{2}\binom{5}{2} = \boxed{30}$

Long Answer: Write your solution in the space below each question.

(a) Compute $25^2, 35^2, 45^2, 55^2$, and 65^2. Can you use this pattern to quickly calculate $75^2, 105^2, 255^2$?

(b) Use the fact that $45^2 = 2025$ to find the prime factorization of 2021.

(c) Is 100100009 prime? Justify your answer (Hint: You may find parts (a) and (b) useful for this problem).

(a) $25^2 = 625, 35^2 = 1225, 45^2 = 2025, 55^2 = 3025, 65^2 = 4225$. Numbers end in "25" and when this is removed, the resulting number is of the form $n(n+1)$.
(b) $2021 = 45^2 - 2^2 = 43 \cdot 47$
(c) This is $10005^2 - 4^2 = 10001 * 10009$ so it's not prime.

use for scratch:

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Short Answer: Write your answer and show your work in the space below each question. Clearly indicate your final answer by drawing a $\boxed{\text{box}}$ around it.

1. If $x^2 - 5x + 2 = 0$ has roots s and t, compute the value of $\frac{s}{t} + \frac{t}{s}$.
   $s + t = 5$ and $st = 2$

$$\begin{aligned}\frac{s}{t} + \frac{t}{s} &= \frac{s^2 + t^2}{st} \\ &= \frac{(s+t)^2 - 2st}{st} \\ &= \boxed{\frac{21}{2}}\end{aligned}$$

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2. Compute the number of intersections of the graphs of $y = |x^2 - 1|$ and $y = x + 1$.
   Sketching reveals 3 solutions. Algebraically,

$$\begin{aligned}(x^2-1)^2 &= (x+1)^2 \\ x^4 - 3x^2 - 2x &= 0 \\ x(x+1)^2(x-2) &= 0\end{aligned}$$

$\boxed{3}$

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3. A triangle has an area of 5 and a perimeter of 10. If the three altitudes have lengths h_1, h_2 and h_3, what is $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3}$?
   Let the sides of the triangle be $x, y,$ and z. Then $5 = \frac{h_1 x}{2} = \frac{h_2 y}{2} = \frac{h_3 z}{2}$
   It follows that $x = \frac{10}{h_1}, y = \frac{10}{h_2}, z = \frac{10}{h_3}$
   From this, $10 = \frac{10}{h_1} + \frac{10}{h_2} + \frac{10}{h_3}$
   So $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \boxed{1}$

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4. List all ordered pairs of positive integers (x, y) that are solutions of $x^2 - y^2 = 51$.
$(x-y)(x+y) = 51$ so $x - y = 1, x + y = 51$ or $x - y = 3, x + y = 17$ from which we get $\boxed{(26, 25), (10, 7)}$.

5. What is the remainder when 2^{2021} is divided by 11?

$$\begin{aligned} 2^{2021} &\equiv 2 \cdot 2^{2020} \mod 11 \\ &\equiv 2 \cdot (2^{10})^{202} \mod 11 \\ &\equiv 2 \cdot 1^{202} \mod 11 \\ &\equiv \boxed{2} \end{aligned}$$

6. Compute the highest power of 9 that divides 2021!.

$$\text{highest exponent of } 3 = \left\lfloor \frac{2021}{3} \right\rfloor + \left\lfloor \frac{2021}{3^2} \right\rfloor + \left\lfloor \frac{2021}{3^3} \right\rfloor + \left\lfloor \frac{2021}{3^4} \right\rfloor + \left\lfloor \frac{2021}{3^5} \right\rfloor + \left\lfloor \frac{2021}{3^6} \right\rfloor$$
$$= 673 + 224 + 74 + 24 + 8 + 2 = 1005$$

Highest exponent of 9 is $\boxed{502}$

Long Answer: Write your solution in the space below each question. Make sure you include sufficient justification.

7. Let $f(n)$ denote how many positive integers with digits 1 and/or 2 have n be the sum of their digits. For example, $f(4) = 5$ because there are 5 such integers (1111, 112, 121, 211, and 22)

   (a) What pattern is made by the values in the sequence $f(1), f(2), f(3), f(4), \cdots$?

   (b) Prove that the pattern holds forever.

(a) 1 (*1*),2(*11,2*),3(*111,12,21*),5(*1111,112,211,121,22*),8(*11111,1112,1121,1211,2111,122,221,212*) - Fibonacci

(b) Each number counted in $f(n)$ ends in a 1 or a 2. If it ends in a 1, the rest of the string adds up to $n-1$. There are $f(n-1)$ such numbers. If it ends in a 2, the rest of the string adds up to $n-2$. There are $f(n-2)$ such numbers. It follows that $f(n) = f(n-1) + f(n-2)$

Use for scratch:

Use for scratch:

Short Answer: Write your answer and show your work in the space below each question. Clearly indicate your final answer by drawing a $\boxed{\text{box}}$ around it.

1. $2\cos x, 3\sin x, 4\tan x$ form a geometric sequence (in that order) with $\sin x \neq 0$. Find $\sin x$

$$(2\cos x)(4\tan x) = (3\sin x)^2 \text{ so } \sin x = \boxed{\frac{8}{9}}$$

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2. In a circle of radius r a chord of length 24 splits the perpendicular diameter into segments whose lengths are in a ratio of 1:9. Compute r.

   Suppose the diameter is split into segments $r - x$ and $r + x$ which are in the ratio $1 : 9$. It follows that $x = \frac{4}{5}r$. The segments then have lengths $\frac{9}{5}r$ and $\frac{1}{5}r$. Using power of a point, $\frac{9}{5} \cdot \frac{1}{5}r^2 = 144$ from which $r = \boxed{20}$

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3. How many of the integers from 1 to 2021, inclusive, can be represented as the difference of two squares?.

   Note that $(a+1)^2 - a^2 = 2a+1$ so all odd numbers are representable. Also, $(a+2)^2 - a^2 = 4(a+1)$ so every multiple of 4 is representable. Odd multiples of 2 are not representable; if they were, $x^2 - y^2 = n = (x - y)(x + y)$ so one of $x - y$ and $x + y$ is even, but $x - y$ and $x + y$ have the same parity - contradiction.
   There are 505 multiples of 4 in the range and 1011 odd numbers, so altogether, we have $\boxed{1516}$

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4. What is the smallest positive power of x that does NOT appear in the expansion of $(1+x+x^3)^{10}$.

We can represent any number of the form $3a+b$ where $a+b \leq 10$. If $a \leq 8$, we can represent all numbers this way and so any number below 26 is representable. 27 and 28 are also representable, but $\boxed{29}$ is not.

5. $\tan A = \dfrac{\cos 2021^\circ + \sin 2021^\circ}{\cos 2021^\circ - \sin 2021^\circ}$ and $0^\circ \leq A < 180^\circ$. Compute A.

$$\begin{aligned} \tan A &= \frac{\frac{1}{\sqrt{2}}\cos 2021^\circ + \frac{1}{\sqrt{2}}\sin 2021^\circ}{\frac{1}{\sqrt{2}}\cos 2021^\circ - \frac{1}{\sqrt{2}}\sin 2021^\circ} \\ &= \frac{\sin(2021^\circ + 45^\circ)}{\cos(2021^\circ + 45^\circ)} \\ &= \tan 2066^\circ \\ &= \tan 266^\circ \\ &= \tan 86^\circ \end{aligned}$$

A = $\boxed{86^\circ}$

6. Compute the remainder when $x^{2021}+1$ is divided by x^2-1.

Set $f(x) = x^{2021}+1$. Then

$$\begin{aligned} f(x) &\equiv (x^2-1)Q(x) + Ax + B \\ f(1) &= A + B \\ \text{so } A + B &= 2 \\ f(-1) &= -A + B \\ \text{so } -A + B &= 0 \end{aligned}$$

We see that $A = B = 1$ so the remainder is $\boxed{x+1}$

Long Answer: Write your solution in the space below each question. Make sure you include sufficient justification.

7. The Fibonacci numbers are defined as $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Show that there exists an integer A such that the sum of any ten consecutive Fibonacci numbers can be represented as A times some Fibonacci number (possibly from the group of ten being summed). Make sure to include the value of A in your answer. You don't have to prove uniqueness.

   Suppose the first two of any 10 consecutive Fibonacci numbers are a and b. The remaining numbers are therefore $a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, 8a+13b, 13a+21b, 21a+34b$.
   The sum is $55a+88b = 11(5a+8b)$. $A = 11$ and $5a+8b$ is the 7th number in the sequence.

I worked independently on this exam.

Name: ______________________