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2018 Summer High School Program Placement Exam for Level A NAME:________________

Multiple Choice: Indicate your answer in the box to the right of each question.

1. A certain person has figured out that he can tile the square shaped floor in his bathroom with square tiles. If 64 tiles are needed to tile the border between the floor and the walls, how many tiles are needed to cover the whole floor, including the border? (A) 128 (B) 225 (C) 289 (D) 900 (E) 1024	C
2. Which of the following fractions is closest to 1? (A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{5}{4}$ (D) $\frac{6}{7}$ (E) $\frac{11}{13}$	D
3. If $2^{2x} = 4$ and $(-2)^{5y} = -32$, then what is the value of $(-2)^{x-y}$? (A) -4 (B) -1 (C) $-\frac{1}{4}$ (D) $\frac{1}{4}$ (E) 1	E
4. If December 3rd of a certain year was a Sunday, what day of the week is January 25th of the next year? (A) Monday (B) Tuesday (C) Wednesday (D) Thursday (E) Friday	D
5. What is the degree measure of the angle between the hour and minute hands on an analog clock at 7:30? (A) 15 (B) 30 (C) 40 (D) 45 (E) 60	D
6. Determine the number of distinct positive factors of 1001. (A) 2 (B) 4 (C) 6 (D) 7 (E) 8	E
7. A fair coin is tossed 5 times. What is the probability that it will land tails at least once? (A) $\frac{1}{32}$ (B) $\frac{5}{32}$ (C) $\frac{13}{16}$ (D) $\frac{27}{32}$ (E) $\frac{31}{32}$	E
8. The degree measure of an interior angle of a regular polygon is a perfect square. How many sides does the polygon have? (A) 8 (B) 10 (C) 12 (D) 18 (E) 24	B

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Short Answer: Write your answer and show your work in the space below each question. Clearly indicate your final answer by drawing a $\boxed{BOX}$ around it.

1. Find all values of x for which the below expression is undefined:

$$\frac{x^2+4}{\sqrt{x^2-4}}$$

$x^2 \leq 4$

$\boxed{-2 \leq x \leq 2}$

2. Find the sum

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\frac{2}{6}+\frac{2}{7}+\frac{3}{4}+\frac{3}{5}+\frac{3}{6}+\frac{3}{7}+\frac{4}{5}+\frac{4}{6}+\frac{4}{7}+\frac{5}{6}+\frac{5}{7}+\frac{6}{7}$$

$\frac{1}{2}+1+\frac{6}{4}+\frac{10}{5}+\frac{15}{6}+\frac{21}{7} = \frac{1}{2}+1+\frac{3}{2}+2+\frac{5}{2}+3$

$= \boxed{10\frac{1}{2}}$

3. What is the last digit of 3^{2018}?

$3^1 \equiv 3 \mod 10$ $\quad 3^2 \equiv 9$ $\quad 3^3 \equiv 7$ $\quad 3^4 \equiv 1$ $\quad 3^5 \equiv 3$ etc

$3^{2016} \equiv 1$ $\quad 3^{2018} \equiv \boxed{9}$

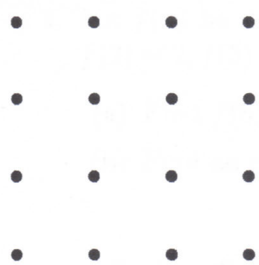
4. In the middle of the Sahara desert there is a tribe of Hungry Sand Eaters. Two Hungry Sand Eaters can eat 10 pounds of sand in 4 days. How much sand will 5 Hungry Sand Eaters eat in 6 days?

2 eat 10 lbs in 4 days
1 eats 5 lbs in 4 days
5 eat $\boxed{37.5\text{ lbs}}$ in 6 days

5. Factor completely: $3x^5 - 6x^3y^2 + 3xy^4$

$3x(x-y)^2(x+y)^2$

6. How many rectangles can be formed with the vertices on the below lattice? [Note: The sides of the rectangles do not necessarily have to be horizontal or vertical.]



parallel to sides:

$${}_4C_2 \times {}_4C_2 = 36$$

tilted: 8

total = $\boxed{44}$

7. If $x + y = 2$ and $x^2 + y^2 = 3$, find the value of $x^3 + y^3$.

$(x+y)^2 = 4$ so $x^2+2xy+y^2=4$

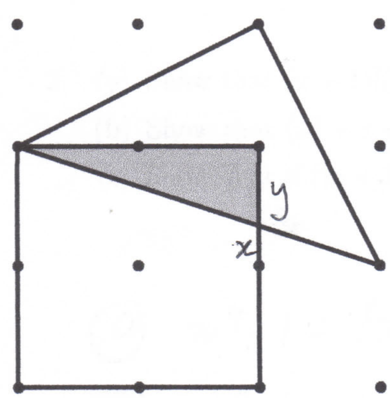
$xy = \frac{1}{2}$

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$

$$= 2(3-\tfrac{1}{2})$$

$$= \boxed{5}$$

8. Find the area of the shaded region:



By similar △s,

$x=\frac{1}{3}$, $y=\frac{2}{3}$

shaded area = $(\frac{2}{3})(2)(\frac{1}{2})$

$= \boxed{3}$

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Long Answer: Write your solution in the space below each question. Make sure you include sufficient justification.

1. Let $f(n)$ be the maximum number of intersections that $n \geq 2$ circles can have. For example, $f(2) = 2$, $f(3) = 6$, and so on.

   (a) Find $f(5)$.

   (b) Find an explicit formula for $f(n)$.

(a) $f(2)=2$, $f(3)=6$

$f(4)=12$, $\boxed{f(5)=20}$ 1

(b) $f(n)=n(n-1)$ each time we add a circle to n circles, we add $2n$ points of intersection

so $2+4+\cancel{4}6+\dots+2(n-1)$ 3

$= n(n-1)$

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2. (a) Show that $(x+1)|(x^4-1)$ (i.e., $(x+1)$ is a factor of x^4-1)

   (b) Show that $(x^2+x+1)|(x^6-1)$

   (c) Show that if $(m+1)|n$, then

$$(1+x+x^2+\ldots+x^m)|(x^n-1)$$

(a) $x^4-1=(x-1)(x+1)(x^2+1)$ 1

(b) $x^6-1=(x^3-1)(x^3+1) = \frac{x^3-1}{x-1}(x-1)(x^3+1)$

$= (x^2+x+1)(x-1)(x^3+1)$ 1

(c) set $n=k(m+1)$ and $y=x^{m+1}$

$1+y+y^2+\dots+y^{k-1} = \frac{y^k-1}{y-1} = \frac{x^{(m+1)k}-1}{y-1}$ 3

$= \frac{x^n-1}{y-1}$

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so $y-1 \mid x^n-1$

i.e. $x^{m+1}-1 \mid x^n-1$ but $x^{m+1}-1=(x-1)(1+x+x^2+\dots+x^m)$

so $1+x+x^2+\dots+x^m \mid (x^n-1)$

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2018 Summer High School Program   Placement Exam for Level B   NAME:________________

Multiple Choice: Indicate your answer in the box to the right of each question.

1. Which of the following is closest to $\frac{1}{2}$? (A) $\frac{2}{3}$ (B) $\frac{7}{13}$ (C) $\frac{9}{19}$ (D) $\frac{10}{21}$ (E) $\frac{23}{48}$	E
2. If $4^{2x} = 4$ and $4^{-5y} = 32$, then what is the value of $(-4)^{x-y}$? (A) -4 (B) -2 (C) $-\frac{1}{4}$ (D) $\frac{1}{4}$ (E) 4	A
3. On an analog clock, how many times do the hour and minute hands coincide between 11:30am on Monday and 11:30am on Tuesday? (A) 12 (B) 13 (C) 22 (D) 23 (E) 24	C
4. If $-1 \leq a \leq 2$ and $-2 \leq b \leq 3$, find the minimum value of ab. (A) -6 (B) -5 (C) -4 (D) -3 (E) -2	C
5. A fair coin is tossed 5 times. What is the probability that it will land tails at least half the time? (A) $\frac{3}{16}$ (B) $\frac{5}{16}$ (C) $\frac{1}{2}$ (D) $\frac{11}{16}$ (E) $\frac{13}{16}$	C
6. Let $f(x) = \lvert 5-2x \rvert$. If the domain of $f(x)$ is $(-3,3]$, what is the range of $f(x)$? (A) $[-1,11)$ (B) $[0,11)$ (C) $[2,8)$ (D) $[1,11)$ (E) None of these	B
7. When $(a+b)^{2018} + (a-b)^{2018}$ is fully expanded and all the like terms are combined, how many terms are there? (A) 1008 (B) 1009 (C) 1010 (D) 2018 (E) 2019	C
8. How many factors does 10! have? (A) 64 (B) 80 (C) 120 (D) 240 (E) 270	E
9. How many x-intercepts does the graph of $\lvert \lvert x-1 \rvert - 2 \rvert + \lvert \lvert y-1 \rvert - 2 \rvert = 1$ have? (A) 0 (B) 1 (C) 2 (D) 3 (E) 4	C

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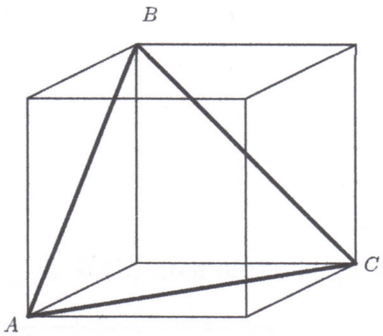
Short Answer: Write your answer and show your work in the space below each question. Clearly indicate your final answer by drawing a $\boxed{BOX}$ around it.

1. If the 3D-shape below is a cube with edge length 1, what is the area of ΔABC?



equilateral Δ, side length $\sqrt{2}$

area $= \frac{\ell^2}{4}\sqrt{3} = \boxed{\frac{\sqrt{3}}{2}}$

2. An analog clock shows 7:45. In how many minutes will the hour and minute hands coincide?

Time is x mins after 8.

minute hand turns through $6x°$

hour " " " $\frac{1}{2}x°$

$240 + \frac{1}{2}x = 6x$ so $x = \frac{480}{11} = 43\frac{7}{11}$

so time elapsed $= \boxed{58\frac{7}{11}}$ or $\boxed{58.7 \text{ approx}}$

3. John is leaving his house in the morning, but his lightbulb broke so he has to select his clothes in complete darkness. He wants to select same color socks and t-shirt. In his bottom drawer, he has 5 pairs each of **unpaired** black, white and brown socks and in the top drawer he has 2 black, 3 white and 4 brown t-shirts. He first takes socks one by one from the bottom drawer and then t-shirts from the top drawer and then walks into another room to put on the clothes. What is the minimum number of items John needs to take from the drawers in order to guarantee that he will have same color socks and t-shirt?

worst case: matching pair of black socks (need at most 4 socks to get this).

get every color-t-shirt: 8

total $= \boxed{12}$

4. Find the largest integer k such that 2^k is a factor of $100 \cdot 101 \cdot \ldots \cdot 200$.

multiples of 2: 100, 102, .. 200 → 51

" 4: 100, 104, .., 200 → 26

" 8: 104, 112, .., 200 → 13

" 16: 112, 128 .. 192 → 6

32: 128, 160, 192 → 3

64: 128, 192 → 2

128: 128 → 1

total $= \boxed{102}$

5. How many 9's are there in the product

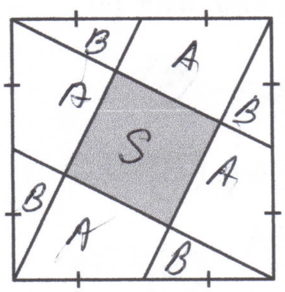
$$\underbrace{999\ldots9}_{1000}\times 9$$

$= 9$ followed by 1000 zeros $- 9$

$= \underbrace{899\ldots9}1$

$\boxed{999}$

6. If the area of the large square is 1, find the area of the shaded region:



$\Delta[A+B] \sim \Delta[A+2B]$

hypotenuse ratio $= \dfrac{1}{\sqrt{1+\left(\frac{1}{2}\right)^2}} = \dfrac{2}{\sqrt{5}}$

so $\dfrac{A+B}{A+2B} = \dfrac{4}{5}$ so $A = 3B$

$4A + 4B + S = 1$

and $A + 2B = \frac{1}{4} \Rightarrow B = \frac{1}{20}, A = \frac{3}{20}$

and $\boxed{S = \frac{1}{5}}$

7. What are the last two digits of 3^{2018}?

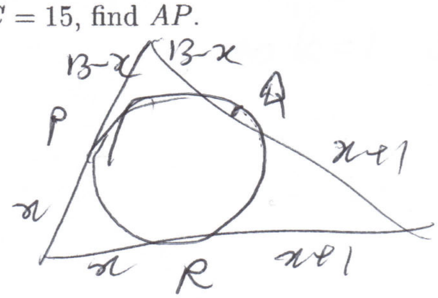
$\phi(100) = \phi(2^2)\phi(5^2) = (2^2-2)(5^2-5)$
$= 40$

$3^{40} \equiv 1 \mod 100$

$3^{2000} \equiv 1 \mod 100$

so $3^{2018} \equiv 3^{18} \equiv (3^6)^3 \equiv 29^3 \equiv 41 \cdot 29 \equiv \boxed{89}$

8. In ΔABC the inscribed circle is tangent to $\overline{AB}, \overline{BC}, \overline{AC}$ at P, Q, R, respectively. If $AB = 13, BC = 14, AC = 15$, find AP.



$2x + 1 = AC = 15$

$\boxed{x = 7}$

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Long Answer: Write your solution in the space below each question. Make sure you include sufficient justification.

1. A k-digit number $\overline{a_k a_{k-1} a_{k-2} \dots a_1 a_0}_n$ is a number base n where each a_i represents a digit base n. For example,

$$1234_{10} = 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0 \qquad 1234_5 = 1 \cdot 5^3 + 2 \cdot 5^2 + 3 \cdot 5^1 + 4 \cdot 5^0$$

and so on. A repunit prime base n is a number of the form $11 \dots 1_n$ that is prime. For example, 11_{10} is a repunit prime base 10, but $11_5 = 6_{10}$ is not a repunit prime base 5.

   (a) Find all $n < 20$ such that 11_n is a repunit prime.

   (b) Find a repunit prime base 3 and a repunit prime base 5.

   (c) Find, with proof, all repunit primes base 4.

(a) need $n+1$ is prime so $\boxed{n=2,4,6,10,12,16,18}$ 1

(b) $1+3+3^2=13$ so $\boxed{111_3}$ 1

$1+5+5^2=31$ so $\boxed{111_5}$ 1

(c) 3

repunit of length $k+1$:

$$111\dots1_4 = 1+4+\dots+4^k$$
$$= \frac{4^{k+1}-1}{3}$$
$$= \frac{(2^{k+1}-1)(2^{k+1}+1)}{3}$$

If k is even, then $2^{k+1}+1 \equiv 0 \bmod 3$

If k is odd, then $2^{k+1}-1 \equiv 0 \bmod 3$

either way, we get 2 factors so one must be 1

i.e. $2^{k+1}+1=3$ or $2^{k+1}-1=3$

$k=0$ or 1 $\qquad$ $k=0$ not possible

so $k=1$ and $\boxed{11_4}$ is the only repunit.

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2018 Summer High School Program Placement Exam for Level C NAME:________________

Multiple Choice: Indicate your answer in the box to the right of each question.

1. If $-1 \leq a < 2$ and $-2 \leq b \leq 3$, find the minimum value of ab. (A) -6 (B) -5 (C) -4 (D) -3 (E) Does not exist	E
2. Find the value of $(x+1)^2(x+5)+(x+3)^2(x+5)+(x+1)^2(x+3)^2$ when $x=-3$. (A) -8 (B) -7 (C) 7 (D) 8 (E) 16	D
3. Find the last three digits of $35,347 \cdot 32,952 \cdot 62,315$. (A) 305 (B) 344 (C) 360 (D) 420 (E) 880	C
4. Points A, B, C, D are in a plane so that $AB = BD = AD = CD = 20$ and $BC = 20\sqrt{2}$. Find $m\angle BCA$. (A) 22.5° (B) 30° (C) 45° (D) 60° (E) 75°	B
5. Find the sum $\cos^2 5° + \cos^2 10° + \cos^2 15° + \ldots + \cos^2 85°$. (A) $\frac{4\sqrt{3}}{2}$ (B) $4\sqrt{2}$ (C) $\frac{17}{2}$ (D) $\frac{17\sqrt{3}}{2}$ (E) $7\sqrt{2}$	C
6. How many factors of 30! are perfect cubes? (A) 270 (B) 300 (C) 405 (D) 540 (E) None of these	A
7. If $\sin\alpha, \sin 2\alpha, \sin^2\alpha$ is a geometric sequence and $0° < \alpha < 90°$, find $\sin\alpha$. (A) $\frac{\sqrt{17}-1}{4}$ (B) $\frac{\sqrt{17}-1}{8}$ (C) $\frac{\sqrt{65}+1}{4}$ (D) $\frac{\sqrt{65}-1}{4}$ (E) $\frac{\sqrt{65}-1}{8}$	E
8. $\overline{AB}$ is a diameter of circle O and chord $\overline{PQ}$ intersects $\overline{AB}$ at X and is perpendicular to it. If $AO = 3PX$, find $\frac{OX}{XB}$. (A) $\frac{3\sqrt{2}-4}{2}$ (B) $\frac{1}{2}$ (C) $\frac{\sqrt{2}}{4}$ (D) $\frac{\sqrt{3}}{3}$ (E) $8+6\sqrt{2}$	B

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Short Answer: Write your answer and show your work in the space below each question. Clearly indicate your final answer by drawing a $\boxed{BOX}$ around it.

1. Three standard fair six-sided dice are rolled simultaneously and the numbers on their top faces are multiplied. What is the probability that the resulting number is not divisible by 6?

$\# a,b,c$ $\quad P(6 \nmid abc) = P(2 \nmid a, 2 \nmid b, 2 \nmid c) + P(3 \nmid a, 3 \nmid b, 3 \nmid c)$

$- P(2,3 \nmid a, 2,3 \nmid b, 2,3 \nmid c)$

$= \left(\frac{1}{2}\right)^3 + \left(\frac{2}{3}\right)^3 - \left(\frac{1}{2}\cdot\frac{2}{3}\right)^3 = \boxed{\frac{83}{216}}$

2. Let $S = \{3, 7, 11, 15, 19, 23, 27\}$. For how many 3-element subsets of S is the sum of their elements divisible by 3?

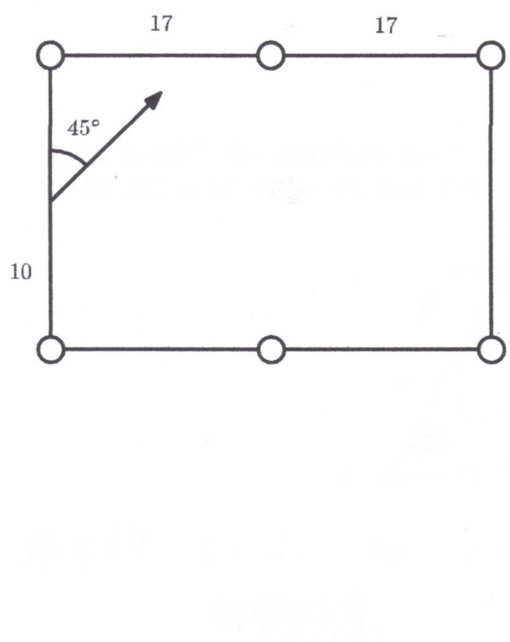
mod 3, there are 3 0's, 2 1's, 2 −1's

case 1: 3 0's 1 way

case 2: 1 '0' and 1 '1' and 1 '−1': $2\times2\times3 = 12$ ways

total $= \boxed{13}$

3. The pool table below has the specified dimensions. A ball is shot at a 45° angle as in the diagram. Which pocket will the ball land in?



BL is (0,0)

Stacking tables horizontally and vertically, eqn of trajectory is $y = x + 10$

Pockets are at $(17n, 20m)$.

solve $20m = 17n + 10$

$10(2m-1) = 17n$

$m = 9$, $n = 10$ is the first solution.

Pocket is at (170, 180)

170 is even, but not a multiple of 4 so pocket is on far right side

180 is an odd multiple of 20 so pocket is on the top

so $\boxed{TR}$

4. In the diagram below, you are allowed to only flip over pairs of adjacent coins. I.e., you may flip only two coins at a time (see diagram). Is it possible to make all the coins Tails? Prove your claim.

(T) (H) → (H) (T)

(T) (H) (H) (T) (H) (H) (H)

$\#H = 7$. This parity does not change with each flip since it increases by 2 or decreases by 2

$\therefore \#H \neq 0$

$\boxed{\text{Impossible}}$

5. When $(a+b+c)^{99} + (a-b+c)^{99}$ is fully expanded and all the like terms are combined, how many terms are there?

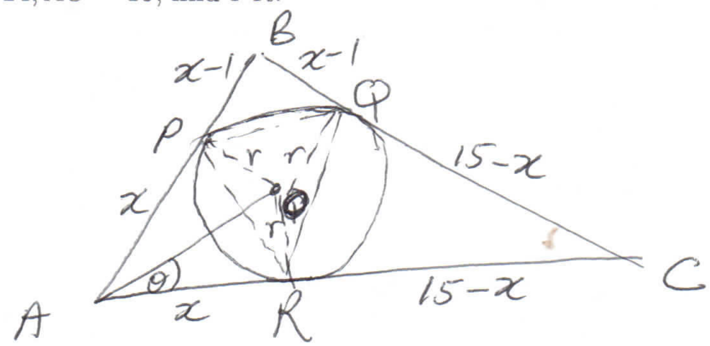
$$\left[(a+c)+b\right]^{99} + \left[(a+c)-b\right]^{99}$$

$$= 2\left[(a+c)^{99} + \binom{99}{97}(a+c)^{97}b^2 + \dots + (a+c)^1 b^{98}\right]$$

$$\#\text{terms} = 100+98+96+\dots+2$$

$$= \boxed{2550}$$

6. In ΔABC the inscribed circle is tangent to $\overline{AB}, \overline{BC}, \overline{AC}$ at P, Q, R, respectively. If $AB = 13, BC = 14, AC = 15$, find PR.



$[ABC] = 84$

Since splitting BC into line segments of lengths 5 and 9 forms two right triangles.

$rs = 84 \quad s = 21 \therefore r = 4$

$2x - 1 = 13 \therefore x = 7$

let $\angle OAR = \theta$

$PR = 2 \cdot 7 \sin\theta$ where $\tan\theta = \frac{4}{7}$ $\quad \sqrt{65}, 4, 7$

$= 2 \cdot 7 \cdot 4/\sqrt{65} = \boxed{\dfrac{56}{\sqrt{65}}}$

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Long Answer: Write your solution in the space below each question. Make sure you include sufficient justification.

1. Find, with proof, all positive integers n such that $3^n - 104$ is a perfect square.

set $3^n - 104 = m^2$ then $3^n - m^2 = 104$

Case 1 $n = 2k$: $(3^k - m)(3^k + m) = 104 = 2 \cdot 52, 4 \cdot 26, 8 \cdot 13, 1 \cdot 104$

$(3^k - m) + (3^k + m) = 2 \cdot 3^k$ so only $2 \cdot 52$ works

and so $k = 3$, $\boxed{n = 6}$

case 2 $n = 2k+1$

mod 4 : $(-1)^{2k+1} \equiv m^2$

so $(-1) \equiv m^2$

$m^2 \equiv 3 \bmod 4$ — impossible

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2. Given two sets A and B, a *bijective correspondence* between them is a rule that pairs every element of A with exactly one element of B and every element of B with exactly one element of A.

   (a) Let $A = \{1, 2, 3, 4\}, B = \{1, 4, 9, 16\}$. A bijective correspondence between A and B is $x \leftrightarrow x^2$. Another bijective correspondence (written explicitly) is $\{(1, 4), (2, 9), (3, 16), (4, 1)\}$. Give an example of another bijective correspondence between A and B.

   (b) Let $X = [1, 2], Y = [1, 4]$. A bijective correspondence between X and Y is $x \leftrightarrow x^2$. Give an example of another bijective correspondence between X and Y.

   (c) Give an example of a bijective correspondence between $X = \{1, 2, 3, \ldots\}$ and $Y = \{7, 8, 9, 10, \ldots\}$.

   (d) Give an example of a bijective correspondence between $X = (0, 1)$ and $Y = (1, \infty)$.

   (e) Give an example of a bijective correspondence between sets $X = [1, 2]$ and $Y = (1, 2]$.

(a) $\{(1,16)\ (2,9)\ (3,4)\ (4,1)\}$

(b) $x \Leftrightarrow 3x-2$

(c) $x \Leftrightarrow x+6$

(d) $x \Leftrightarrow \frac{1}{x}$

(e) Example: $x \Leftrightarrow \frac{3}{2}$ if $x = 1$

$x \Leftrightarrow 1 + \frac{1}{2^{n+2}}$ if $x = \frac{1}{2^n}$

$x \Leftrightarrow x$ otherwise

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