Fall

- Pythagorean Theorem
- Pigeonhole Principle
- Recursion
- Combinatorics - introduction
- Combinatorics - Cookies and Kids
- Probability
- Induction

Spring

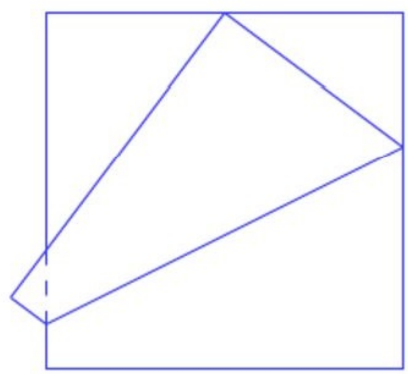
- Divisibility, Congruence, and Modular Arithmetic.
- Euclidean Algorithm
- Roots, Squares, Cubes
- Inequalities
- TART (Triangle Area Ratio Theorem)
- Mass Points
- Invariants

Possible Summer Topics

- Expected Value and Random Walks
- Continued Fractions
- Polygons Surrounding a Point
- Graph Theory
- Circles
- Vieta's Formula

Sample Problems

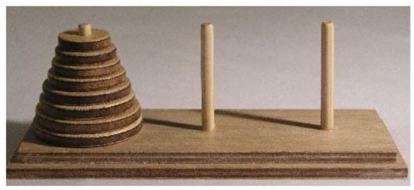
1. (Pythagorean Theorem) A square piece of paper, eight inches on a side, is folded so that one corner meets the midpoint of the opposite side, as shown. The paper is then unfolded. What is the length of the crease?



2. (Pigeonhole Principle). Show that, in a class with 25 students, there are at least two students with the same number of friends in the class. (Friendship is always mutual).
3. (Recursion). Seven waiters stand in a line, all facing the same direction. Each waiter has a tray of food which he holds in either his left hand, his right hand, or straight out in front of him in both hands. However, if a waiter holds a tray in his right hand, then the waiter immediately to his right must not hold a tray in her left hand. In how many ways can the waiters hold their trays?
4. (Combinatorics). Ben and Jon are engaged in an epic chess match. The first person to win seven games will win the match. How many different sequences of games won and lost are possible? (For example, if J stands for a win for Jon and B stands for a win for Ben, one possibility is JJBJBJJJBBJ.)
5. (Cookies and Kids). How many six digit positive integers have the property that the sum of their digits is 9?
6. (Probability). Three cards are selected from a deck. What is the probability that they are all different suits?
7. (Probability). (AMC 2010A No.18) Bernardo randomly picks three distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks three distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

8. (Probability). Jack and Jill toss a coin to determine who will have to fetch a pail of water. Jack, however, has a lucky coin which comes up heads, on average, three out of every five tosses. Jack and Jill agree to toss the coin until one of them has won two more tosses than the other. (Jack is heads.) What is the probability that Jill will have to fetch the water?

9. (Induction). Prove that the Tower of Hanoi puzzle (with n disks) can be solved in $2^n - 1$ moves, and this is the smallest number of moves possible.



10. (Induction). Prove that among any 2^{n+1} natural numbers there are 2^n numbers whose sum is a multiple of 2^n.

11. (Divisibility). What is the prime factorization of the sum of all of the divisors of 600?

12. (Divisibility). There are digits a and b such that the positive integer $ab123456789$ is a multiple of 99. Find a and b.

13. (Divisibility). How many divisors of 90,000 have exactly 12 factors?

14. (Divisibility). Show that the equation $y^2 = 2x^3 + 5$ has no integer solutions.

15. (Euclidean Algorithm). Farmer Jon has 100 goats, each worth \$121 and farmer Becky has 100 pigs, each worth \$105. Jon owes Becky \$1. To settle the debt, he gives her some goats and she gives him some pigs. How many goats and pigs were exchanged?

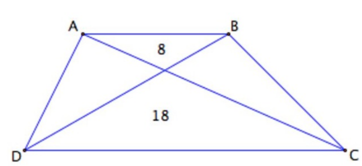
16. (Roots, Squares, and Cubes). Simplify $3\sqrt{29 + 12\sqrt{5}} - 2\sqrt{49 + 12\sqrt{5}}$

17. (Roots, Squares, and Cubes). Find two four digit numbers whose product is $4^8 + 6^8 + 9^8$

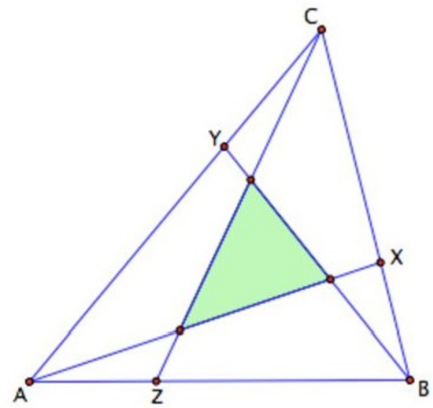
18. (Inequalities.) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$, for all real numbers $a, b, andc$.

19. (Inequalities.). A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs \$2 per foot, while the fence for the other three sides costs \$1 per foot. How much of each type of fence will he have to buy in order to keep expenses to a minimum? What is the minimum expense?

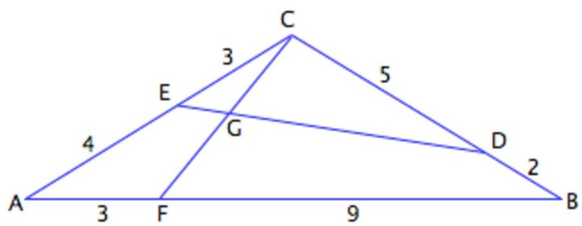
20. (TART). ABCD is a trapezoid and the bottom and top triangles have areas of 18 and 8, respectively. What is the area of the entire trapezoid?



21. (TART). $\frac{AZ}{ZB} = \frac{BX}{XC} = \frac{CY}{YA} = \frac{1}{2}$.
Find the ratio of the area of the shaded triangle to the area of $\triangle ABC$



22. (Mass Points). In $\triangle ABC$ points D, E, and F divide the sides BC, AC, and AB in the ratios $\frac{BD}{DC} = \frac{2}{5}$, $\frac{AE}{EC} = \frac{4}{3}$, and $\frac{AF}{FB} = \frac{3}{9}$. Use mass points to find the ratios $\frac{EG}{GD}$ and $\frac{CG}{GF}$.



23. (Invariants). n cards are labeled 1 through n, where n is a positive integer. The cards are then shuffled. If the top card is labeled k, then the order of the top k cards is reversed. This last step is repeated indefinitely. Prove that eventually the top card will be labeled 1.

24. (Expected Value and Random Walks). A gecko and an anti-gecko take a random walk on the faces of a cube, starting on opposite faces. How long, on average, will it be before the geckos walk onto the same face of the cube and annihilate each other?

25. (Continued Fractions). Find the continued fraction for $\sqrt{3}$.

26. (Polygons Surrounding a Point). Show that three regular polygons, with a, b, and c sides, respectively, can be fit together with no overlap and no empty space if and only if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$$