Year 1: Fall

- Expected Value and Random Walks
- Hard Combinatorics
- Power of a Point
- Roots of Unity, DeMoivre's Theorem, and Polynomials
- Geometric Constructions

Year 1: Spring

- Recursion and Sterling Numbers
- Induction
- AM-GM Inequality
- Cyclic Quadrilaterals
- Radical Axis

Year 2: Fall

- Cookies and Kids
- Principle of Inclusion and Exclusion
- Inversion in a Circle
- Rearrangement Inequality

Year 2: Spring

- Geometric Optimization
- Generating Functions
- Telescoping Sums and Products
- Arithmetic Functions
- Areas of Equilateral Triangles and the Isodynamic Point

The Summer

- Cauchy-Schwarz Inequality, Vectors, and Equations of Lines and Planes
- Congruence, Roots of Unity, Finite Fields, and Quadratic Reciprocity
- Ceva and Menelaus, Mass Points, and Barycentric Coordinates

Sample Problems

1. (Hard Combinatorics) A penguin takes a random walk on a long ladder, starting at the intersection of one of the rungs and one of the sides. Every second she moves, with equal probability, either to the other side of the ladder (on the same rung) or up or down one rung (on the same side). However, she only moves up on one side of the ladder, and she only moves down on the other side. Find the probability that, after one minute, the penguin is back where she started.

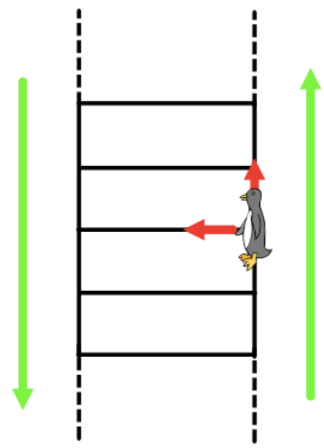
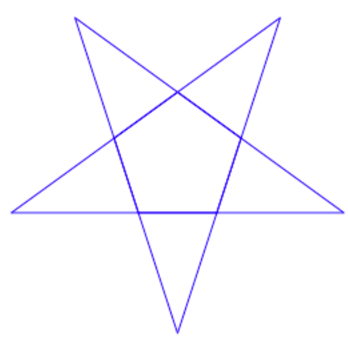


Figure 1: A penguin on a ladder

2. (Inclusion-Exclusion) Four couples are sitting in a row. In how many ways can they be seated so that no person is sitting next to their partner?

3. (Recursion) An elephant seal takes a walk on the ten vertices of a pentagram, waddling from one vertex to any adjacent vertex. If the elephant seal starts on one of the five outer vertices, in how many ways can she take a 15 step walk?



4. (Sterling Numbers) The *Stirling number of the first kind*, written $\begin{bmatrix} n \\ k \end{bmatrix}$ and pronounced "n

cycle k", is the number of permutations, with precisely k cycles, of a set of n elements. Find and prove a recursive formula for $\begin{bmatrix} n \\ k \end{bmatrix}$.

5. (Generating Functions) Let t_n be the number of ways that the $3 \times 2n$ rectangle can be tiled by 2×1 and 1×2 rectangles. (Let $t_0 = 0$.) Find the generating function for the sequence $t_0, t_1, t_2, \ldots$. Use this to find a non-recursive formula for the t_n.

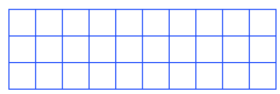


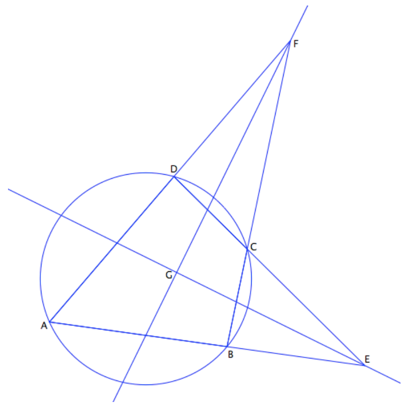
Figure 2: A 3×10 rectangle can be tiled in t_5 ways.

6. (Geometric Constructions) Given three concentric circles, construct an equilateral triangle with one vertex on each of the given circles.

7. (Roots of Unity, DeMoivre's Theorem, and Polynomials) Find the remainder when $x^{2015} - 9$ is divided by $(x^2 + 1)(x^2 + x + 1)$.

8. (Roots of Unity, DeMoivre's Theorem, and Polynomials) Given a quadrilateral, consider the four squares, each of which lies outside of the quadrilateral and shares a side with the quadrilateral. Prove that the two segments formed by joining the centers of opposite squares are congruent and perpendicular.

9. (Cyclic Quadrilaterals) Given the cyclic quadrilateral ABCD, with AB and DC extended to meet at E and AD and BC extended to meet at F, let EG and FG be the bisectors of angles E and F, respectively. Prove that EG and FG are perpendicular.



10. (Power of a Point - AIME II 2015 #11) The circumcircle of acute $\triangle ABC$ has center O. The line passing through point O perpendicular to $\overline{OB}$ intersects lines AB and BC at P and Q, respectively. Also $AB = 5$, $BC = 4$, $BQ = 4.5$, and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. (Radical Axis - AIME 1 2016 #15) Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

12. (Telescoping Sums) Let F_n be the n^{th} *Fibonacci* number, defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$. Evaluate $\sum_{k=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}}$.

13. (Expected Value and Random Walks) A gecko and an anti-gecko take a random walk on the faces of a cube, starting on opposite faces. How long, on average, will it be before the geckos walk onto the same face of the cube and annihilate each other?

14. (Expected Value and Random Walks - 2016 AIME I #13) Freddy the frog is jumping around the coordinate plane searching for a river, which lies on the horizontal line $y = 24$. A fence is located on the horizontal line $y = 0$. On each jump Freddy randomly chooses a direction parallel to one of the coordinate axes and jumps one unit in that direction. When he is at a point where $y = 0$, with equal likelihoods he chooses one of three directions where he either jumps parallel to the fence or jumps away from the fence, but he never chooses the direction that would have him cross over the fence to where $y < 0$. Freddy starts his search at the point $(0, 21)$ and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river.

15. (AM-GM Inequality) If $a, b, c, d > 0$ then $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$.

16. (AM-GM Inequality) Find the dimensions of the cylinder of maximal volume inscribed in a right circular cone with base radius R and height H.

17. (Rearrangement Inequality) Prove **Nesbitt's inequality**: if a, b, and c are positive, then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

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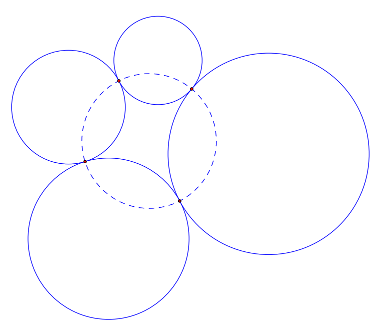
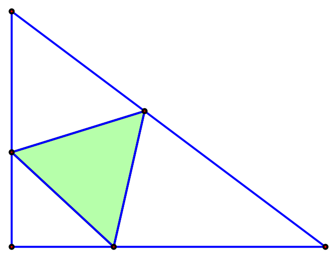
18. (Inversion) Given four circles which are pairwise tangent, prove that the four points of tangency are concyclic.



19. (AIME DAY - 2017 AIME I #12) Call a set S product-free if there do not exist $a, b, c \in S$ (not necessarily distinct) such that $ab = c$. For example, the empty set and the set $\{16, 20\}$ are product-free, whereas the sets $\{4, 16\}$ and $\{2, 8, 16\}$ are not product-free. Find the number of product-free subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

20. (Geometric Optimization - 2017 AIME I #15) Find the area of the smallest equilateral triangle with one vertex on each of the sides of a triangle with sides of length 3, 4, and 5.(Can you

construct this triangle, given the right triangle?)



21. (Geometric Optimization - Fagnano's Problem) Find the triangle of smallest perimeter among all triangles with one vertex on each of the sides of a given $\triangle ABC$.

22. (Arithmetic Functions) The *Euler phi* function is defined as follows. $\phi(n)$ is the number of positive integers less than or equal to n which are relatively prime to n. So, for example, $\phi(1) = 1$, $\phi(2) = 1$, $\phi(4) = 2$, $\phi(10) = 4$, and $\phi(19) = 18$. Prove that $\displaystyle\sum_{d|n} \phi(d) = n$.