PROBLEM SOLVING VS ANSWER FINDING

CHRIS JEUELL
NOTES TRANSCRIBED BY ALISON AUN AND ANDREW YANG

Problem solving vs answer finding: what are some differences between these two activities? Some possibilities include: answer finding has a straightforward objective (namely, find the answer), while problem solving could be deeper and more exploratory. Answer finding can be mechanical, while problem solving not so much. Perhaps the ultimate example of answer finding (as opposed to problem solving) is the Countdown round in Mathcounts, where the only thing that matters is finding the answer as quickly as possible.

As an example of problem solving, in this talk we will examine a question from geometry and see how it is possible to approach it from as many as eight different ways. Material from this talk was inspired by the article "Problem-solving vs. answer-finding", by John Staib, in Two-Year College Mathematics Readings, MAA 1981, pp 221-227.

1. The Problem

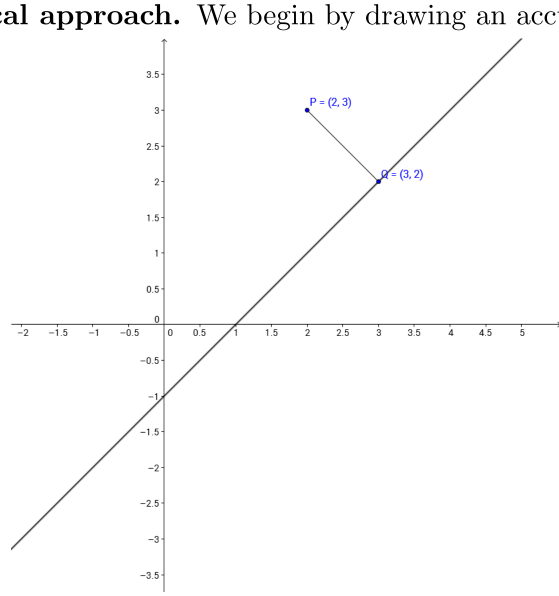
Find the distance from the point $P(2,3)$ to the line $\ell : y = x - 1$.

Before we jump into finding solutions, let's make sure we know precisely what the problem is asking us to solve. There are many points P' on the line ℓ, so there are many distances $|PP'|$. What this question asks for is the minimum possible value of $|PP'|$. Notice that this is identical to the length of the segment obtained by dropping a perpendicular from P to ℓ. (Exercise: why is this true?)

2. Solutions

We will now look at several different ways to solve the problem. A general formula exists which provides the answer to this question, but we won't use it, and as a matter of fact some of these methods can be used to derive that formula.

Solution 1: a classical approach. We begin by drawing an accurate picture.



Let Q be the point on ℓ which is closest to P. We can solve this problem by computing the equation for the line $\overleftrightarrow{PQ}$ and then finding the point of intersection, which is Q. Since ℓ has slope 1, the line $\overleftrightarrow{PQ}$ has slope $-1/1 = -1$, since $\ell \perp \overleftrightarrow{PQ}$. The point-slope equation, applied to the point $P = (2,3)$, tells us that this line has equation $(y-3) = -1(x-2)$, or $y = -x+5$.

We simultaneously solve the system $y = -x+5$ and $y = x-1$ to find the coordinates of the point Q. Algebra shows that $Q = (3,2)$. Therefore the distance from P to the line ℓ is the distance from P to Q, which is given by the distance formula

$$d(P,Q) = \sqrt{(P_x - Q_x)^2 + (P_y - Q_y)^2} = \sqrt{(2-3)^2 + (3-2)^2} = \sqrt{2}.$$

Solution 2: A moving point and slope. Let's try solving this problem using a *parameterization* for ℓ. Let P' be an arbitrary point on ℓ with x-coordinate equal to t. Then $P' = (t, t-1)$. Conversely, any point of the form $(t, t-1)$ lies on ℓ. We want to find the value of t which makes the line segment connecting P and $(t, t-1)$ perpendicular to ℓ. Recall that from the first solution we know this segment should have slope -1. If $Q = (t, t-1)$, then the slope of PQ is given by

$$\frac{3-(t-1)}{2-t} = \frac{4-t}{2-t}.$$

Setting this equal to -1 and solving for t gives $4 - t = t - 2$, or $t = 3$, which tells us that the nearest point Q has coordinates $(3,2)$. The rest of the solution proceeds as in Solution 1.

Solution 3: A moving point and distance. This solution is similar to the previous solution. Like before, we parameterize a point P' on ℓ with $P' = (t, t-1)$. The distance $|PP'|$ is given by the formula

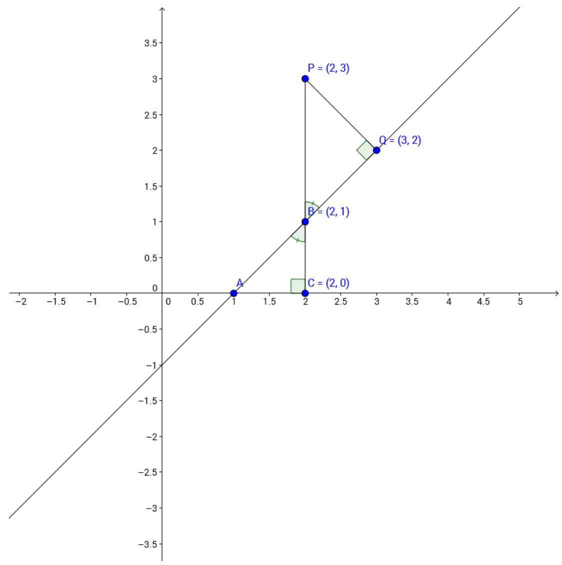
$$\sqrt{(2-t)^2 + (3-(t-1))^2} = \sqrt{(2-t)^2 + (4-t)^2} = \sqrt{2t^2 - 12t + 20} = \sqrt{2}\sqrt{t^2 - 6t + 10}.$$

Recall that we want to minimize this value. Because $f(x) = \sqrt{x}$ is an increasing function of x, minimizing this value is equivalent to minimizing the value of the quadratic $t^2 - 6t + 10$. We can complete the square on this quadratic:

$$t^2 - 6t + 10 = (t-3)^2 + 1.$$

This is minimized when the squared expression is equal to 0; ie, when $t = 3$. Therefore the nearest point is $(3,2)$, just as before. As a matter of fact, notice that we can directly compute the distance since we already have a formula for the distance in terms of t: plugging in $t = 3$ into $\sqrt{2}\sqrt{t^2 - 6t + 10}$ gives $\sqrt{2}$.

Solution 4: Similar triangles.



Drop a line segment from P to the x-axis. Let $C = (2, 0)$ be the point at the other end of P on this segment, and let $B = (2, 1)$ be the intersection of this line segment and ℓ. Finally, let $A = (1, 0)$ be the intersection of ℓ with the x-axis. Then notice that $\triangle ABC$ is similar to $\triangle PBQ$. Because $\angle ABC$ and $\angle PBQ$ are vertical angles, these two angles are equal. Also, the angles at C and Q are right angles, so we know that $\triangle ABC$ is similar to $\triangle PBQ$ instead of $\triangle QBP$.

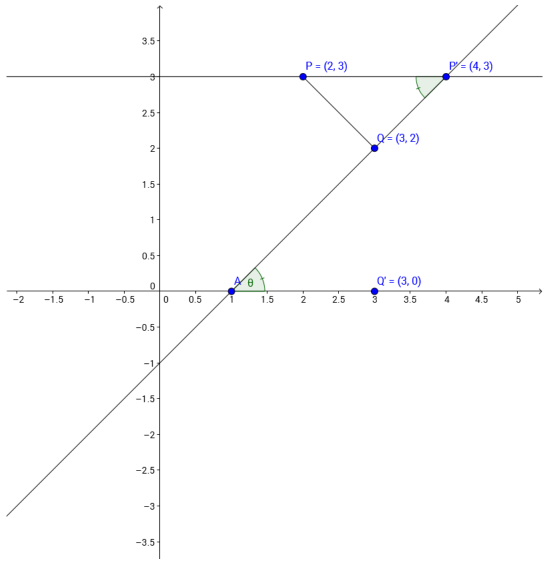
We know what the lengths of the sides in $\triangle ABC$ are: $|AC| = |BC| = 1$, so by the Pythagorean Theorem $|AB| = \sqrt{2}$. Also, we know what the length of PB is; it is $|PB| = 2$. Therefore, similar triangles tells us

$$\frac{|PB|}{|AB|} = \frac{|PQ|}{|AC|}.$$

We want to solve for $|PQ|$, and fortunately we know the values of all the other segments in this equation. Plugging those values in gives

$$\frac{2}{\sqrt{2}} = \frac{|PQ|}{1} \implies |PQ| = \sqrt{2}.$$

Solution 5: Trigonometry.

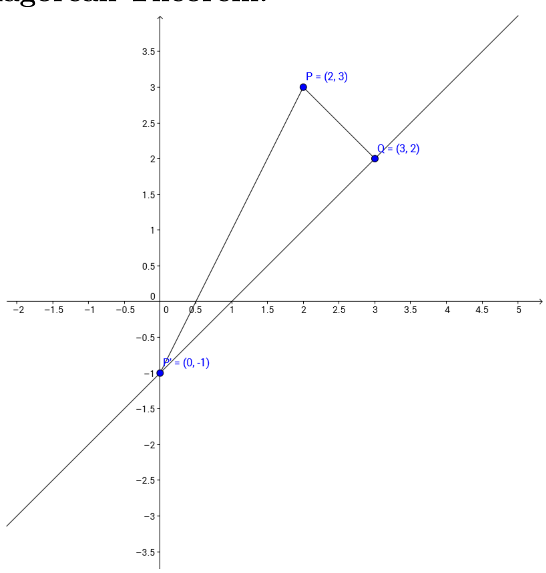


In the figure above, let θ be the measure of the angle formed by the line ℓ and the (positive) x-axis. Then $\tan\theta$ is equal to the slope of this line: for example, if we drop a line segment from Q to the x-axis which intersects the x-axis at a point Q', then $\triangle AQ'Q$ forms a right triangle, and $\tan\theta = |QQ'|/|AQ'|$. However, this is just the slope of ℓ.

With this in mind, since we already know ℓ has slope 1, this means that $\tan\theta = 1$, or $\theta = 45°$.

Now draw the horizontal line which passes through $P = (2, 3)$ and let its intersection with ℓ be the point $P' = (4, 3)$. Since $\angle QAQ'$ and $\angle QP'P$ are alternate angles, they have equal measure. Therefore $\angle QP'P$ has measure $45°$. Since we know that $|P'P| = 2$ and $P'P$ is the hypotenuse of the isoceles right triangle $QP'P$, this means $|QP| = \sqrt{2}$.

Solution 6: the Pythagorean Theorem.



Let $P' = (t, t-1)$ be an arbitrary point on ℓ. Also, let $Q = (x, x-1)$ be the coordinates of Q. We want to solve for x. Since $\triangle PQP'$ is a right triangle, we can apply the Pythagorean theorem:

$$|PP'|^2 = |PQ|^2 + |P'Q|^2,$$

or substituting in the values for the coordinates,

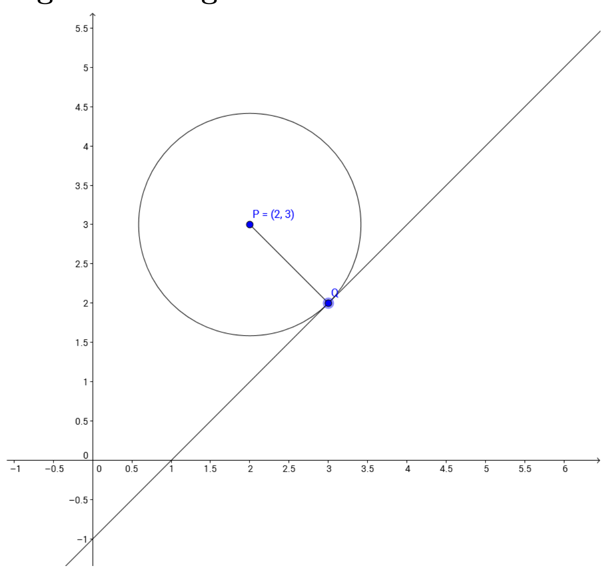
$$(2-t)^2 + (4-t)^2 = (2-x)^2 + (4-x)^2 + 2(x-t)^2.$$

Since t can be any number except x, let's arbitrarily pick $t = 0$. Then

$$4 + 16 = 2x^2 - 12x + 20 + 2x^2 \iff 4x^2 - 12x = 0 \iff 4x(x-3) = 0.$$

This has solutions $x = 0, 3$. However, since $t = 0$ can't be equal to x, this means that $x = 3$ is the x-coordinate of Q, so $Q = (3, 2)$. The rest of the solution proceeds like other solutions we have already seen.

Solution 7: A growing circle tangent to ℓ.



In this solution, let C be a circle centered at $P = (2, 3)$. Imagine C growing from a small circle until it first intersects ℓ. Because this is a point of tangency, the point of intersection is Q.

If C has radius r, then C is defined by the equation

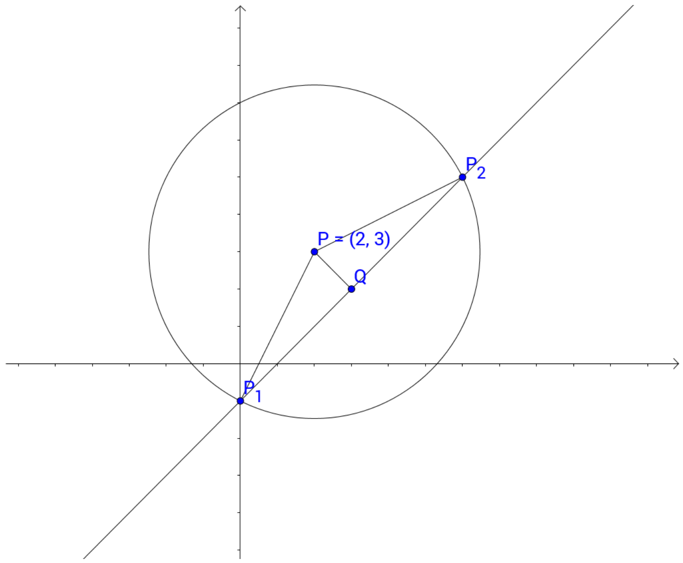
$$(x-2)^2 + (y-3)^2 = r^2.$$

On the other hand, we want to find the smallest value of r such that a point $(x, x-1)$ lies on C: this is the same as finding the smallest circle C centered at P which intersects ℓ. Plugging in $y = x - 1$ into the equation for this circle, we find

$$(x-2)^2 + (x-4)^2 = r^2.$$

In other words, we want to find the smallest value of r for which this has a solution, which can be solved by minimizing the value of $(x-2)^2 + (x-4)^2 = 2x^2 - 12x + 20$. But this is the same algebraic expression we minimized in Solution 3, and we found the minimum value to be 2, so $r = \sqrt{2}$ is the smallest radius of C which intersects ℓ, and for this value of r, the intersection is the unique point of tangency Q. Hence $|PQ| = r = \sqrt{2}$.

Solution 8: A growing circle with chord.



The setup for this solution is similar to the previous solution, except this time we let the circle grow past the point of tangency. In this case, C will intersect ℓ at two points we call P_1 and P_2. Notice that the point Q whose coordinates we are looking for is the midpoint of the line segment $\overline{P_1P_2}$. (This is because $\overleftrightarrow{PQ}$ is the perpindicular bisector of $\overline{P_1P_2}$.)

How does this help us? Since C is any triangle which intersects ℓ in two points, we can arbitrarily select P_1 to be any point on C. For numerical simplicity, set $P_1 = (0, -1)$. Then C has radius $d(P_1, P) = \sqrt{(2-0)^2 + (3-(-1))^2} = \sqrt{20}$. Therefore C has equation given by

$$(x-2)^2 + (y-3)^2 = 20.$$

We want to compute the other intersection of C with ℓ. Since all points on ℓ have the form $(x, x-1)$, let's plug $y = x - 1$ into the equation for C:

$$(x-2)^2 + (x-1-3)^2 = 20 \iff x^2 - 4x + 4 + x^2 - 8x + 16 = 20 \iff 2x^2 - 12x = 0.$$

This has solutions $x = 0, 6$. Since $x = 0$ corresponds to the point $P_1 = (0, -1)$, this means $P_2 = (6, 5)$. Since Q is the midpoint of $\overline{P_1P_2}$, $Q = (3, 2)$.