The Art of Mathematical Modeling

For these exercises, please also refer to the **Algorithm for Constructing a Mathematical Model**. I encourage the use of difference equations rather than differential equations to make the following problems accessible to middle school and high school aged students.

Simple Machines: Pendulums

Consider a simple pendulum consisting of a straight, fixed length rod with small, uniform mass topped by a weight (or bob).

1. Formulate a list of questions we can ask about the physical qualities of the pendulum system. We'll focus on just one of these qualities for now.
2. Write a list of the physical qualities of the pendulum system that affect its motion. Don't over think this - we'll throw out the negligible effects and unrealistic assumptions later. It's always good to start with a brain dump.
3. Throw out the physical qualities that can be neglected or that are unrealistic for a **simple** system. Assign variables and parameters to the key physical quantities that are left.
4. Let's investigate the period of the pendulum, or the time it takes for the pendulum to return to its initial position. By balancing the dimensions of the period with that of the key variables, determine the approximate form of the function that describes the period of the pendulum.
5. By equating the forces acting on the pendulum bob, derive the equation of motion for the angle swept out by the pendulum bob over time.
6. How would you then write the equation for a damped pendulum?
7. How might we experimentally test the validity of this model? Be precise.

Complex Systems: Projectiles

Consider a projectile launched near the surface of the Earth. We'd like to determine qualities about the motion of this projectile - whether its position relative to the Earth's surface, the velocity required to escape Earth's gravitational pull, or velocity required to reach a particular target or to impart a particular impact. This type of problem is originally intended for investigation using Vector Calculus. It can easily be modified to study projectile motion in one dimension by having the projectile launched straight upwards, making it more appropriate for a high school aged crowd.

1. Write a list of physical qualities of the projectile system that affect its motion.
2. Trash the physical qualities and assumptions that make the system too complex, but not so much that it makes the problem unrealistic. Assign variables and parameters to what is left over.

3. Write an abstract vector function (using the variables and parameters you determined) that represents the position of the projectile in 2-D space.

4. Using Newton's Laws, write a vector function describing the acceleration of the projectile.

5. Integrate the vector function you found twice to determine the position of the projectile in time.

6. How would you determine the maximum height the projectile reaches for a given angle?

Complex Systems: Chemical Flow

Consider the following scenario: A child is rushed to the emergency room after having consumed one 5mg hydrocodone tablet, which their parents had in the medicine cabinet after recent surgery. You are to consider the following known information about the child and the drug ingested and determine if the child is in danger. We'll then discuss how to save the child from lethal poisoning. [1]

We know the following:

- The child has approximately 2 liters of blood.
- The child is admitted to the hospital 1 hour after consuming the drug. This means that the hydrocodone has had time to pass from the stomach to the intestines, so it is too late to eliminate the drug via vomiting.
- Hydrocodone has an absorption half-life of 0.5 hours and an elimination half-life of 3.8 hours. This means that it takes 0.5 hours for hyrdocodone to transition from the gastrointestinal tract to the bloodstream, and 3.8 hours to get from the bloodstream to the excretory system.
- Hydrocodone can be lethally toxic at a concentration of approximately 0.5 mg/L. [2]
- Like most oral drugs, hydrocodone is absorbed into the bloodstream at a rate proportional to the amount present in the gastrointestinal tract (stomach and intestines), and is eliminated from the bloodstream at a rate proportional to the amount present in the bloodstream.
- The antidote for hydrocodone overdose is Naloxone, which counteracts hydrocodone's possibly lethal depression of the central nervous system and respiratory system. One can also increase the rate of drug elimination by administering charcoal and laxatives.

We want to address the following:

[1] Hydrocodone is a potent opioid analgesic. In 2012, the second highest cause of poisoning of children under the age of 6 was pain medication.

[2] Approximated from a collection of studies.

1. In order to determine whether the patient is in danger and how to effectively treat them, we would like to know about the **amount** of hydrocodone in the patient's bloodstream over time. Really what you want to know is the **concentration** of hydrocodone in the bloodstream over time. How can you convert **amount** of hydrocodone in the bloodstream to **concentration** of hydrocodone in the bloodstream?

2. We have identified three physiological **compartments** through which the hydrocodone must flow once entering the body. What are these three compartments? Draw a simple, labeled diagram of the flow between these compartments. Do we want to track the amount of hydrocodone in all of these compartments? Explain.

3. What physical factors effect the movement of the drug through the three physiological compartments? Translate the most important factors of the system into parameters and variables. This should include a dependent variable for the amount of hydrocodone in each compartment you want to track.

4. How many dependent variables are needed to describe the drug in its various compartments?

5. Since we want to model the state (both the location and the amount) of the drug over time, it makes sense to write equations for the drug's rate of change. Note that we should have one equation for every dependent variable in order to be able to solve. We can write an abstract continuity equation to describe the flow rate of a system:

   $$\text{(flow rate of change of stuff in system)} = \text{(flow rate in)} - \text{(flow rate out)}$$

   We want to write one of these equations for each dependent variable. We'll start by assuming that we know how time is measured. We have an **initial** time, or the time the drug is ingested, which we call t_0. Since all of our data is given in hours, we will measure the time in hours. After t_0, time proceeds in equally-spaced steps as follows:

   $$t_0,\, t_1,\, t_2,\, ...,\, t_{i-1},\, t_i,\, t_{i+1},\, ...,\, t_{n-1},\, t_n$$

   where t_n is a self-imposed maximum for use when we put our equations into computer software. Rather than deriving **differential equations**, which would make this problem inaccessible to students without a knowledge of Calculus, we are going to derive and analyze **difference equations**.

   If we define $x(t_i)$ as the amount of drug in compartment X at time t_i, then

   $$\text{rate of change of drug amount } x = \frac{\text{change in amount}}{\text{change in time}} = \frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}$$

   Note that $t_{i+1} - t_i$ should remain the same, regardless of the value of t, so you might want to give it a different name. What then are the equations for the drug flux in each compartment? Use the variables and parameters you determined in part 3. Make sure that the units on both sides always balance.

6. Now we want to use our data to find parameters and solve these equations. For all of your equations, solve for your dependent variable evaluated at t_{i+1} - so $x(t_{i+1})$ on the left hand side, $x(t_i)$ on the right hand side. We now have equations that are **recursive**, or defined in terms of previous values. For now, let's call our unknown parameters α_1 and α_2.

7. Using the given absorption half life, determine the value of α_1, or the coefficient of your variable representing amount of hydrocodone in the gastrointestinal tract. You can assume for simplicity that $t_{i+1} - t_i = 0.1 \quad \text{hours}$.

8. We can now numerically solve the equation for amount of hyrdocodone in the gastrointestinal tract. Generate and plot the solution using your choice of computer software (MATLAB, Python, Excel, R, Sage, Maple, Mathematica). What type of **continuous** function does this plot resemble? How would you re-write the solution to this equation in terms of this function?

9. Now let's consider the elimination half life so we can solve for the amount of hydrocodone in the bloodstream. In order to find the coefficient of the variable representing amount of hydrocodone in the bloodstream, we can assume that at some future time, there is 0.02 mg of hydrocodone in the bloodstream with no more flow in. This means we can temporarily modify our equation for bloodstream hydrocodone amount to have zero influx. **This is only a temporary assumption** - it will be used solely for the purpose of obtaining the second parameter.

   Observing the equation in this new form, note that it closely resembles the equation we had for the amount of hydrocodone in the gastrointestinal tract. Using the re-written solution for part 8 as a guide, what should be the form of our approximated solution for amount of hydrocodone in the bloodstream? Use this form to find the second unknown parameter, α_2. Note that this form is only an approximation of the solution after a certain amount of time has elapsed - it is **NOT** the actual solution!

10. Now that we have α_2, generate and plot the solution (using your results from part 6) for the amount of hydrocodone in the bloodstream over top of the amount of hydrocodone in the gastrointestinal tract.

11. From your graph, determine the amount of hydrocodone in the child's bloodstream at the time of admission to the hospital.

12. Determine the amount of hydrocodone in the bloodstream that would constitute a lethal level. Does the amount of hydrocodone upon the child's admission to the hospital constitute a lethal amount?

13. What's the largest amount of the drug that is ever in the patient's bloodstream, and at what time does it occur?

Other problems to investigate:

- What then would be the lethal amount of hydrocodone in the bloodstream for an adult with 6 liters of blood?
- To save the child, we can give him/her charcoal and/or laxatives, which increases the rate at which the drug is eliminated from the bloodstream. How do we determine whether the increased rate at which the drug is eliminated is sufficient to save the patient? [3]

References and Further Reading

- COMAP, The Consortium for Mathematics and Its Applications - www.comap.com. Filled with wonderful applied resources for math teachers of all levels.
- The UMAP Journal - A Quarterly publication from COMAP. Contains insightful articles about math education as well as high school and undergraduate level research. Excellent subject material for research projects and applied problems. Also publishes the MCM results.
- HiMAP and UMAP Modules - Problem sets and research problems published in the UMAP Journal.
- Jodye I. Selco, Janet L. Beery. *Saving a Drug Poisoning Victim*, ILAP Modules: Tools for Teaching 2000, 31–46.
- The Illustrative Mathematics Project - www.illustrativemathematics.org. Exploration of the Common Core Standards through applied illustrative problems submitted and edited by math professionals.
- Bouncing Balls and Geometric Series - http://www.maa.org/external_archive/joma/Volume7/Styer/index.html - An excellent interactive physics and mathematics lab, perfect for the classroom.
- TranscendEd Consulting - www.transcendEdconsulting.com. My company, providing quantitative tutoring services and educational consulting services.

[3] We can also administer Naloxone intravenously, but this drug acts on cells in the brain to counteract the effects of the hydrocodone, and so is unable to be tested with this particular compartmental model.