<u>Euler's Formula</u>

Of the many formulas Euler discovered, one of the most admired is:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The formula provides an elegant method for evaluating complicated integrals and summing series; it also provides an elegant proof of De Moivre's Theorem and can be used to derive many trigonometric formulas. Distinguished theoretical physicist Richard Feynman described it as "one of the most remarkable, almost astonishing, formulas in all of mathematics".

<u>History</u>

The formula emerged after attempts by a number of mathematicians to evaluate complex logarithms. Johann Bernoulli noted that

$$\int \frac{1}{1+x^2}\,dx = \frac{1}{2i}\int\left(\frac{1}{1-ix} - \frac{1}{1+ix}\right)dx$$

leading to a complex log expression. Roger Cotes, noted that

$$\ln(cosx + isinx) = ix$$

but he missed the fact the equation was true modulo multiples of $2\pi i$. After correspondence with Bernoulli, Euler turned his attention to the exponential function instead and thereby obtained the now famous formula. The proof is straightforward:

From Taylor's Theorem:

$$e^{i\theta} = 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots,$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right)$$

$$= \cos\theta + i\sin\theta$$

Since the cosine and sine series are known to converge for all θ, the whole series converges.

As an aside, setting $\theta = \pi$ gives us $e^{i\pi} + 1 = 0$, an identity that links the fundamental numbers $e, i, \pi, 1, 0$. Also, geometrically, multiplication by $e^{i\theta}$ corresponds to rotation by θ in the Argand Diagram.

<u>De Moivre's Theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$</u>

This is often proved in many textbooks, if at all, by induction. Euler's formula yields a simple proof:

$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$, showing that De Moivre's Theorem is also valid for non-integer exponents.

Integration

Noting, for example, that $\int e^x \cos x \, dx = Re\left(\int e^{(1+i)x} \, dx \right)$

Euler's formula can be used to simplify many trigonometric integrals.

Textbooks

Further Mathematics by R.I. Porter (for UK schools). Freely available online at:

http://archive.org/stream/FurtherMathematics/PorterFurtherMathematics (p137-139)

Problems

1. Choosing the appropriate value of θ, calculate a possible value for

$$\sqrt{\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)}$$ in the form $a + bi$ where a and b are real numbers.

2. Find all solutions of $x^3 = 1$.

3. Calculate i^i

4. By looking at $e^{i(A+B)}$ and $e^{i(A-B)}$ obtain the addition formulas for

$$\cos(A + B), \sin(A + B), \cos(A - B) \text{ and } \sin(A - B)$$

5. Prove that the infinite series

$$1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta + \frac{1}{16}\cos 4\theta + \cdots = \frac{4 - 2\cos\theta}{5 - 4\cos\theta}$$

6. Use Euler's formula to determine $\int e^x \cos x \, dx$ and $\int e^x \sin x \, dx$

7. Use Euler's formula to show that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, and thereby prove that

$$\int \sin^4 x \, dx = \frac{1}{8}\left(\frac{1}{4}\sin 4x - 2\sin 2x + 3x\right) + \text{ constant}$$

8. Alternative proof of Euler's Formula: A complex number e^{ix} can be written in its polar form $r(\cos\theta + i\sin\theta)$ where r and θ are functions of x

   (a) Differentiating both sides, show that $\frac{dr}{dx} = 0$ and $\frac{d\theta}{dx} = 1$
   (b) Prove that $e^{ix} = \cos x + i\sin x$