## 21-Nim

Here's a great warm-up game that is similar to Nim. There is one pile of 21 counters. On your turn, you can take 1, 2 or 3 counters. You win if you take the last counter. In a *losing* position, the next player will lose if her opponent plays correctly. Challenge: find and $\boxed{\text{box}}$ the set of *losing positions*!

$\boxed{0}$ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

## Winning and Losing Positions

Each game we consider will play out in a finite number of moves, resulting in a win or a loss. If the next player to move can force a win through strategic play, we call this game position **W**inning. Otherwise the opponent can force a win, and the next player is in a **L**osing position.

From a Losing position, every possible move leads to a Winning position (for the opponent). From a Winning position, there is at least one possible strategic move to a Losing position. To know the set of *losing positions* is to know the winning strategy!

## The Game of Nim

The game of Nim seems to originate in China as *Jianshizi* (picking stones). The name *Nim* and the winning strategy were first described and proved in a math research paper by Charles Bouton in 1901. Nim also makes an appearance in *Last Year at Marienbad*, a 1961 French film.

Nim is a take-away game starting from a certain number of counters grouped into piles. Players alternate, and a legal move consists of removing one or more counters from any one pile. The player to remove the last counter from the last pile wins the game. Equivalently, the player left with no legal move loses.

## Nim Challenges

Which of the following are *losing positions*? Notation: 3 denotes a pile of of 3 counters, 3 + 2 denotes two piles, one with 3 counters and one with 2 counters. The 0 is the empty Nim game with zero counters. Since the empty game is a losing position, I've boxed that to get you started.

$$
\begin{array}{ccccccccccc}
5 & & 4+1 & & 3+2 & & 2+3 & & 1+4 & & 5 \\
& 4 & & 3+1 & & 2+2 & & 1+3 & & 4 & \\
& & 3 & & 2+1 & & 1+2 & & 3 & & \\
& & & 2 & & 1+1 & & 2 & & & \\
& & & & 1 & & 1 & & & & \\
& & & & & \boxed{0} & & & & &
\end{array}
$$

1. Make a conjecture about which two pile Nim games are *losing positions*. Prove it!
2. The three pile Nim game $1 + 7 + 7$ is a winning position. What is the strategic move?
3. Explain why there can be at most one value of $c$ for which $1 + 2 + c$ is a losing position.
4. Find the value of $c$ so that $1 + 4 + c$ is a losing position. How about $1 + 2a + c$?
5. If $a + b + c$ is a losing position, then $2a + 2b + 2c$ is also a losing position. And vice-versa?

## The Game of Jim (Japheth's Nim)

Jim was invented by Japheth Wood in 2011. A Jim game starts with several rows of solid tokens $\bullet$ and empty tokens $\circ$. Players alternate moves: select a row, and change one or more tokens (solid to empty or empty to solid). **RULE:** The first token-change from the left *must* be solid to empty (but does not need to be the leftmost solid token). The last player with a legal move wins. Equivalently, if you only see empty tokens, then you've just lost!

### Jim Challenges

Which of the following Jim games are *losing positions*?

1. One row Jim: $\circ \circ \circ$ (all empty tokens). One row Jim: $\bullet \circ \bullet$.
2. Two row Jim: $\bullet \circ \bullet \mid \bullet \circ \bullet$. Two row Jim: $\circ \circ \bullet \mid \bullet \circ \bullet$.
3. Make a conjecture about which one row and two row Jim games are *losing positions*.
4. (Difficult) Explain why this three row Jim game is a *losing position*: $\circ \bullet \bullet \mid \bullet \circ \bullet \mid \bullet \bullet \circ$.

### Nim and Jim

1. Explain why, once a row of Jim is changed, no sequence of legal moves will ever return it to its original sequence of solid and empty tokens.
2. What is the *maximum* number of legal moves that will change row $\bullet \circ \bullet$ to row $\circ \circ \circ$?
3. Can you find a one-pile Nim game that has the same maximum number of legal moves?
4. Make a conjecture about the connection between Nim and Jim games.

### NYMC Teachers' Math Circles

The NYMC Teachers' Math Circle is a professional development opportunity for teachers seeking to deepen their math content knowledge, develop their problem-solving skills, discuss mathematics, work on problems, and have dinner together in a growing community of math teachers. Please join us for another session, and invite your colleagues!

### For Further Reading

Charles L. Bouton, *Nim, A Game with a Complete Mathematical Theory*, The Annals of Mathematics Vol. 3, No. 1/4 (1901 to 1902), 35–39.

Elwyn R. Berlekamp John H. Conway and Richard K. Guy, *Winning Ways for your mathematical plays,* Academic Press, London, 1982.

Julius G. Baron *The Game of Nim-A Heuristic Approach*, Mathematics Magazine, Vol. 47, No.1 (1974) 23–28.