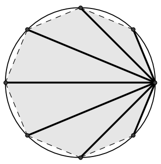
The Circle Chord Problem



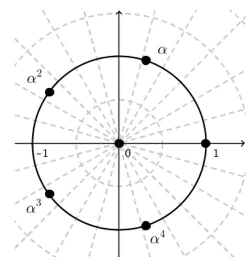
A regular polygon of n sides is inscribed in a unit circle. Chords are drawn from one vertex to the other $n-1$ vertices. What is the product of the lengths of the chords?

Why in the world are we multiplying the chord-lengths?

Solving challenging problems is a tradition in mathematics going back to the beginning. Work on unsolved problems is essential to mathematical research, and expands human knowledge. Many mathematicians enjoy encapsulating intricate work in the form of a challenging problem with a surprising result.

Complex Roots of Unity

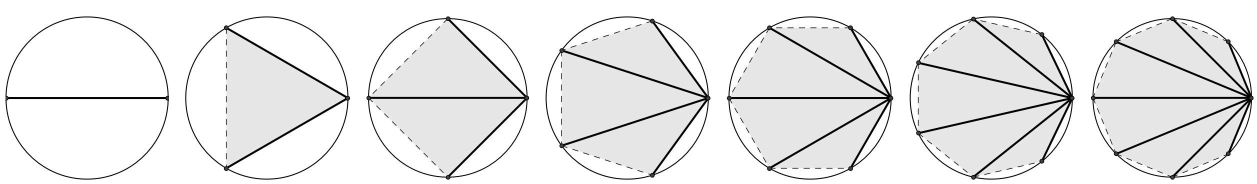
The roots of the equation $z^n - 1 = 0$ are n complex numbers $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$, equally-spaced around the unit circle. Thus, they form the vertices of a regular n-gon. The product of the chord lengths is $P = |(1-\alpha)| \cdot |(1-\alpha^2)| \cdots |(1-\alpha^{n-1})|$.



The identity $(z-1)(z-\alpha)(z-\alpha^2)\ldots(z-\alpha^{n-1}) = z^n - 1$ yields
$(z-\alpha)(z-\alpha^2)\ldots(z-\alpha^{n-1}) = z^{n-1} + \ldots + z^2 + z + 1$.
Substituting $z = 1$ shows that the product of the chord lengths is just n.

Problems

1. Compute the exact chord lengths of an equilateral triangle, square, regular pentagon, hexagon, and octagon inscribed in a unit circle.



2. Compute the approximate chord lengths of a regular heptagon inscribed in a unit circle.

3. Use trigonometry to express the chord lengths of a regular n-gon inscribed in a unit circle.

4. The stretched chord problem: Draw the chords of a regular n-gon inscribed in a unit circle and then stretch the figure vertically by a factor of $\sqrt{5}$. Now multiply the lengths of the stretched chords. Express the product as a function of n.

Japheth Wood
<japheth@nymathcircle.org>

Ben Blum-Smith
<ben@blumsmithmath.org>